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ABSTRACT

This document consists entirely of materials presented at the Third International Conference held at Warwick University, England, from July 9 through 14, 1979. The items are collected and listed alphabetically by author. An opening address is included, and a total of 45 papers are presented, covering a wide variety of topics. (MP)

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PROCEEDINGS OF THE
THIRD INTERNATIONAL CONFERENCE
FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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WARWICK UNIVERSITY 9 - 14 JULY 1979

edited by David Tall

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Josette Adda

U.E.R. de DIDACTIQUE des DISCIPLINES
UNIVERSITE PARIS VII

"... ETC ..."

Many people are still saying that mathematical language is always unambiguous but didacticians know that many errors come from misunderstandings about ambiguous expressions. Moreover, there are often expressions implicit, unsaid. I will focus here on problems involved in the use of "etc", "and so on", and "...".

I. Some strange questions

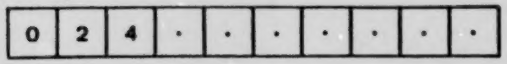
Everybody knows the psychological tests of the type "1,3,5... continue". These tests claim to "measure the intelligence". Anybody who would "continue" by "0,0,0" or by "1,3,5,1,3,5,1,3,5" or by his birthdate would probably be considered feeble-minded or abnormal (1).

Most mathematicians object to such exercises, as they may lead a child to believe that a function on a set is uniquely determined as soon as one knows its values for some points of the set.

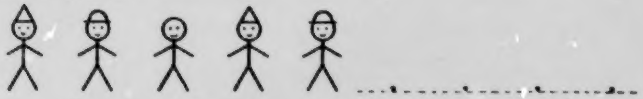
The most disquieting development is the influence recently exerted by certain trends in psychology over many mathematics teachers. As a result, one now finds in texbooks many questions of the above type, presented as "mathematical questions".

Here are some examples of exercises drawn from a texbook $[M_1]$ intended for pupils age 6-7:

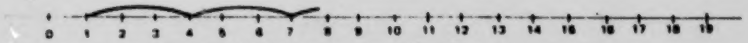
1 - Observe et complète



5 - Observe et continue :



6 -



Observe et continue ; écris la liste des nombres trouvés :
1 ; 4 ; 7 ;

These questions are very selective because they are of the kind:"find what I am thinking of"and they can even be dishonest, like this one, found in the game section of a newspaper:

"Complete 13 - 1 - 9 - 19 - 15 - ?"

The answer was "14" in order to have the word "MAISON" !!

The reason I regard this as dishonest is that in every body's mind, the context was numbers, while the authors of the game moved on to another level, as in puns. Thus it seems to me that we have here sources of misunderstandings very close to those caused by the confusions that I studied in $[A_1]$.

- Literally "et cetera" means "and the other things" and the problem is that the referential domain of these other things is left unsaid.

II Mathematical uses

1. Extrapolations - Interpolations.

Can we absolutely say that the ability to guess what the teacher has in mind never is a good sign?

The questions asked as above are certainly stupid but what about the questions as " Propose some interesting manners to continue" ?

All mathematicians who protest as I said will occasionally write:
"Consider $a_0 + a_1 x + \dots + a_n x^n$" or even, having presented a sequence (u_n), "consider the sum $u_1 + u_3 + u_5 + \dots$" and is it not true that every reader will understand the same thing?

Interpolations and extrapolations from some cases quoted are very frequent. In some cases, such as the preceding examples, one can rely on the consensus, because they involve, in whole or part, the sequence of natural numbers. On the other hand, some cases require a more or less developed mathematical sense, in order to imagine a more or less interesting embedding.
We often speak of interpolations or extrapolations which are "obvious", "trivial", "canonical" , by "good functions"; these terms are all soft and imprecise, however they are in constant use by mathematicians and almost always there is large agreement about their meaning, yet is it necessarily so among students or pupils?

When a mathematician conjectures that a property verified by certain elements of a set is true everywhere on this set, he knows very well that he has proved nothing but he has perhaps made progress in his research with the help of this reasoning by analogy.

Every student, for example, has occasions to conjecture an induction relation from the values taken by some of the first terms of a sequence.

---

(1) In a selection test intended for unemployed men, we have witnessed the following astute answer given grade 0 : "19,35,64" - the question being : "Extend "1,3,6,10,☐,☐,☐" The expected answer was "15,21,28"!

2. Paradigm - Generalization - Introduction.

When there is an associative binary law $\ast$ on a set, we often see the definition of a n-ary law by: "We can definite $a_1 \ast a_2 \ast a_3$ as $a_1 \ast (a_2 \ast a_3)$ (which is $(a_1 \ast a_2) \ast a_3$)) and so on for $a_1 \ast a_2 \ast a_3 \ldots \ast a_n$ for every n".

On many occasions two or three particular "paradigmatic" cases are studied and one says " etc " in order to express that the property is "general", is satisfied everywhere where the same conditions are.

(See, for example, the proof by Euclide of the 20th proposition: the sequence of prime numbers is infinite.)

This process of generalization is sometimes finite, sometimes infinite, (according to the set where it is applied. When it is denumerably infinite, we are often dealing with an intuitive process of mathematical induction because "etc" is used when the speaker feels that the proof he has just given to go from 1 to 2 and from 2 to 3 for example, is paradigmatic and could be done "in the same way" to go from any n to n + 1 (1).

H. Freudenthal tells us in "Mathematics as an educational task" [F_2] how a young girl discovered the process of numeration: after having written numbers and numbers one after the other , suddenly "at 1024 she obstinately refused to go further. "So it goes on", she said, does it not? ". And H Freudenthal comments: "But whatever you think about it, the exclamation "so it goes on" is mathematics? it is the first mathematics mankind produced and individuals are producing: It is great and important mathematics, it is first and last mathematics; it is the loftiest and the most profound mathematics. Is it ridiculous to stress it? There are people who do not know this - that is my excuse".

3. Taking limits

Well, that was mathematics; but why do so many people be suspicious before that kind of behaviour? It seems to me that it is because we cannot know if it is (like probably in this case) deep understanding or mere intuition.

And for intuition there is often risk of confusions: We can see very often confusions between "f (n) for all the n " and " $\lim_{n\to\infty} f(n)$".

We can quote for instance the classical "proof" that "the length of hypotenuse is equal to the sum of the lengths of the sides of the right angle" : you divide the hypotenuse in 2,4,8,16,... equal parts and obtain a sequence of saws (lines with sides parallels to the sides of the right angle) which all have the same length and when you continue dividing the hypotenuse, these saws are nearer and nearer the hypotenuse so the hypotenuse is pretended to have this length too.

In an exam (see [A_2] where students had to establish a property of $\bigcup_{n\in\mathbb{N}} X_n$, many people established only this property for $\bigcup_{p<n} X_p$, for every n.

E. Fishbein (see [F_1]) made the following experiment:

> "Pupils from grade 5 to 6 (aged 12 to 15) had been presented with the following: " C is an arbitrary point somewhere on a segment A B. We divide and subdivide the segment A B by two, by four, by eight. We continue dividing in the same manner.
>
> Question :"Will we arrive at a situation such that one of the points of division will coincide with point C?" The great majority of the subjects (about 80%) affirmed that the point would be reached".

When I tried to test similar questions with French students asking:

Question A:

> " Soit un point M quelconque sur un segment A B donné. Est-ce que, si l'on coupe A B en deux, puis chacune des parties en deux, etc..., on atteindra le point M ?"

Question B:

> "Sur un segment A B donné, on considère l'ensemble E des points suivants : le milieu de A B, les milieux des deux segments déterminés par le point précédent, les milieux des quatre segments obtenus et ainsi de suite. Est-ce-que, pour tout point M de A B, il existe un point de l'ensemble E confondu avec M ?"

I noticed not only that I had more "no" than "yes" (it is normal because these students were more advanced in mathematics than pupils tested by E. Fishbein) but that there was not a significant difference between question A and question B (which I have thought better), but many students asked "What do you mean by "etc" ?, it is that expression which could have been understood as including accumulation points.

---

(1) Notice that sometimes you can make a mistake when not having seen that it is not paradigmatic, for example:

"Proposition: $n^2 - n + 41$ is a prime number for every natural n".

"Proof:

$1^2 - 1+41 = 41$; $2^2 - 2+41 = 43$; $3^2 - 3+41 = 47$; $4^2 - 4+41 = 53$;

$5^2 - 5+41 = 61$; $6^2 - 6+41 = 71$; etc...

It is not true because at least $41^2-41+41$ is not a prime number! But this can be a trap for many students, however, as a good student wrote when I proposed it: "Three dots have never been a proof". But I am less sure that he will never use a proof of this kind, in other circumstances!

III. Evolution of the mathematical style.

J.J. Rousseau wrote in "Eléments de géométrie" (1801) this strange definition:

> 5. Si une ligne AB (*Fig.* 1.) tourne au tour du point A, de maniere que son extrémité aille de B en C, de C en D, etc. le point B décrira en tournant, une ligne courbe BCDFGLB.
>
> 6. Cette ligne courbe s'appele *Circonférence* du cercle. Le *Cercle* est proprement l'espace renfermé par cette circonférence. Le point A s'appele *Centre* du cercle. Toutes les lignes droites AC, AD, AF, etc tirées du centre à la circonférence, s'appellent *Rayons*. Un *Diamètre* est une ligne droite qui passe par le centre, et qui est terminée de part et d'autre à la circonférence; la ligne DAL, par exemple, est un diamètre. Un *Arc* est une partie de la circonférence, comme par exemple FG.

and we can read few pages later:

> *Tous les rayons d'un même cercle sont égaux entr'eux.*
>
> DÉMONSTRATION. Pour former le cercle BCDFGLB, (*Fig.* 1.) nous avons dit qu'il falloit faire tourner la ligne AB au tour du point A. Or, lorsque le point B en tournant est sur le point C, toute la ligne AB se trouve sur la ligne AC; sans quoi deux lignes droites renfermeroient un espace, ce qui est impossible; le rayon AC est donc égal à la ligne AB. On prouvera de la même maniere que les rayons AD, AF, AG, etc. sont aussi égaux à la ligne AB; ils sont donc tous égaux entr'eux.

You can think perhaps that this is a caricature. After all, J.J. Rousseau was not a mathematician! But the analysis of many ancient texts of mathematicians presents very often a great use of "...etc...".

Consider for example the famous great theorem of D'Alembert, (text of 1746 [D1]) about the polynomial equations: in 14 pages, you can find 14 times "&c" and one time "& ainsi de suite" (without noticing the frequent use of points in expressions as " $x^m + a x^{m-1} + b x^{m-2} \dots + f x + g$ "). The most surprising is that 5 times "&c" appears in the short sentence " Donc &c" at the end of a proof, just before " C.Q.F.D.".

This use of " Donc &c" to resume a part considered as too easy to be written at the end of a proof, is not a peculiarity of D'Alembert's style

You can meet it very frequently in ancient texts as for example Arnaud ("Nouveaux éléments de géométrie MDCLXVII) or Miss Agnesi ("Traité élémentaires de calcul différentiel et intégral - traduits en MDCCLXXV).

In ancient texts, we often find also a use of "&c" inside the expression of numbers:

as " $\frac{1}{3} = 0,333$ etc...

$\frac{2}{3} = 0,66666$, etc." (Page 34 - Cours élémentaire et complet de mathématiques pures - La Caille - Paris - An IIIe).

We can also find it inside formulas as " $\Phi(x) = a_0 + a_1 x + a_2 x^2 + \&c\dots$ " (in Cours d'analyse de l'Ecole Royale Polytechnique - Cauchy - 1821)

In Cramer (Introduction à l'analyse des lignes courbes algébriques - MDCCL), page 666 at the 3rd line, we find:

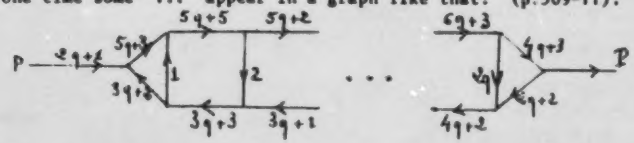
" $x^n - [1]\,x^{n-1} + [1^2]\,x^{n-2} - [1^3]\,x^{n-3} \dots\dots [1^n] = 0$ "

so he uses the "...", but at the 8th line in the same page:

"ab + ac + ad + &c. + bc + bd + &c. + cd + &c."

and so we arrive, page 679, to "&c. &c.".!!

- On the contrary, Bourbaki's style is very different: let us make a sounding taking for example the volume "resultats de topologie", we never find "etc" nor "ainsi de suite" and we find "..." only at the last page with the notation " $x_1 \dots x_n \to e(h_1 x_1 + h_2 x_2 + \dots + h_m x_m)$ "
- Let us consider, from this point of view, a recent volume of Bourbaki's Seminar taken at random (the one of november 1977, containing 6 lectures): You find neither "etc" nor "ainsi de suite". You find only some "..." inside formulas, representing elements of sequences (finite or infinite ones) and one time some "..." appear in a graph like that: (p.509-11).



Another time (p. 510-07), you find some "..." in the expression of a matrix like this:

$$\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

This absence of "etc" is not special to the "Bourbaki's spirit";

For instance ,I have also considered in the same manner the "Compte-rendus de l'Académie des Sciences" (a session at random: the 16^{th} May 1978- p. 803-846). These texts are, by convention, very concise, so you can find many manners to express implicit things, like "il s'ensuit facilement que...", "en multipliant par une fonction convenable", but not even one time there is. never "etc" nor "ainsi de suite".

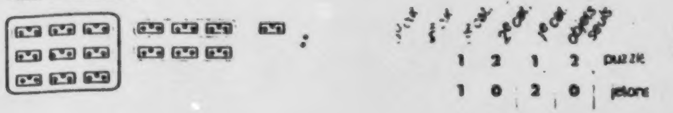
However, we can see many times in these notes some "..." in expressions like " $n = 1,2,\ldots$ " or " $\Lambda_n(\delta_1,\ldots,\delta_n,p)$ ". So it is used for parts of $\mathbb{N}$ or sequences indexed by naturals and nothing else, in particular it is never, in these texts, used to express implicit parts of reasonings.

But this use stays in many texbooks for pupils, for example in a recent textbook $[T_1]$ for children (10-11 years old) we can find these presentations for numeration and division:

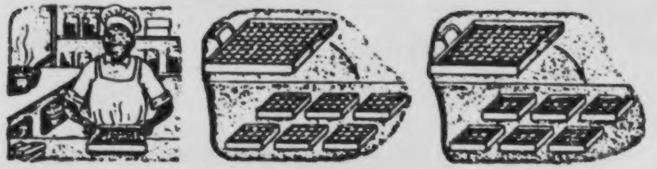
- **Groupements trois par trois**

Combien de pièces pour ce puzzle?
Pour les dénombrer, groupons-les selon une autre règle que celle des groupements dix par dix.

Utilisons un système de boîtes de différentes tailles
— dans chaque petite boîte (boîte de 1ʳᵉ catégorie) on met trois pièces,
— dès que trois boîtes sont remplies nous les plaçons dans une boîte plus grande (boîte de 2ᵉ catégorie),
— dès que trois boîtes de la 2ᵉ taille sont remplies nous les plaçons dans une boîte encore plus grande (boîte de 3ᵉ catégorie), et ainsi de suite.
Voici le résultat :



44. DIVISION ENTIÈRE - PROPRIÉTÉS



Dans une confiserie, un employé garnit des boîtes de caramels

Il prend un 1ᵉʳ plateau
100 caramels au lait
Il garnit 6 boîtes

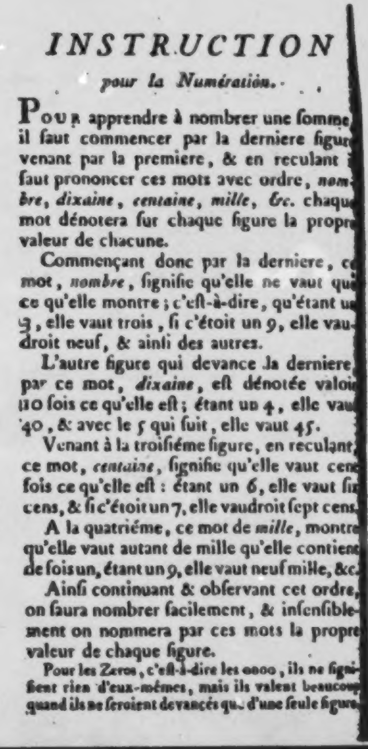
2ᵉ plateau
100 caramels au chocolat
Il garnit encore 6 boîtes

$100 = (6 \times 16) + 4$ et $4 < 6$
16 caramels par boîte
il reste 4 caramels

Au total
200 caramels ; 12 boîtes
$200 = (12 \times 16) + 8$ et $8 < 12$,
16 caramels par boîte
il reste 8 caramels

Pour un 3ᵉ plateau de 100 caramels au café... même répartition, et ainsi de suite !

There are some drawings and "concrete" situations but the difficulty to understand the general method through the little number of cases considered is the same as in the ancient texts as in Barrême (1988) $[B_1]$.



INSTRUCTION

pour la Numération.

Pour apprendre à nombrer une ſomme,
il faut commencer par la derniere figu
venant par la premiere, & en reculant
faut prononcer ces mots avec ordre, *nom-*
bre, *dixaine*, *centaine*, *mille*, *&c.* chaqu
mot dénotera ſur chaque figure la propr
valeur de chacune.

Commençant donc par la derniere, c
mot, *nombre*, ſignifie qu'elle ne vaut qu
ce qu'elle montre ; c'eſt-à-dire, qu'étant u
3, elle vaut trois, ſi c'étoit un 9, elle vau-
droit neuf, & ainſi des autres.

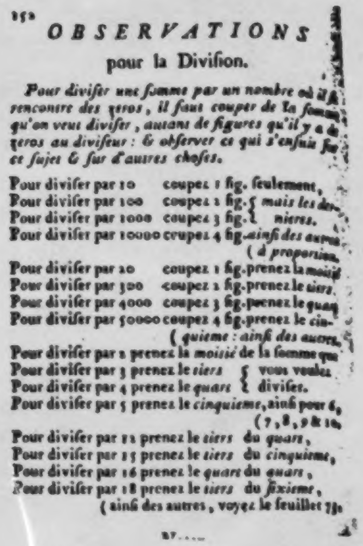
L'autre figure qui devance la derniere
par ce mot, *dixaine*, eſt dénotée valoi
10 fois ce qu'elle eſt ; étant un 4, elle va
40, & avec le 5 qui ſuit, elle vaut 45.

Venant à la troiſiéme figure, en reculant
ce mot, *centaine*, ſignifie qu'elle vaut cen
fois ce qu'elle eſt : étant un 6, elle vaut ſi
cens, & ſi c'étoit un 7, elle vaudroit ſept cens

A la quatriéme, ce mot de *mille*, montr
qu'elle vaut autant de mille qu'elle contien
de fois un, étant un 9, elle vaut neuf mille, &c

Ainſi continuant & obſervant cet ordre,
on ſaura nombrer facilement, & inſenſible-
ment on nommera par ces mots la propre
valeur de chaque figure.

Pour les Zeros, c'eſt-à-dire les 0000, ils ne ſigni-
fient rien d'eux-mêmes, mais ils valent beaucou
quand ils ne ſeroient devancés que d'une ſeule figure



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OBSERVATIONS

pour la Diviſion.

Pour diviſer une ſomme par un nombre où il ſe
rencontre des zeros, il faut couper de la ſomme
qu'on veut diviſer, autant de figures qu'il y a de
zeros au diviſeur : & obſerver ce qui s'enſuit ſur
ce ſujet & ſur d'autres choſes.

Pour diviſer par 10 coupez 1 fig. ſeulement,
Pour diviſer par 100 coupez 2 fig. { *mais les der-*
Pour diviſer par 1000 coupez 3 fig. { *nieres.*
Pour diviſer par 10000 coupez 4 fig. *ainſi des autres*
(*à proportion.*

Pour diviſer par 20 coupez 1 fig. prenez la *moitié*
Pour diviſer par 300 coupez 2 fig. prenez le *tiers.*
Pour diviſer par 4000 coupez 3 fig. prenez le *quar*
Pour diviſer par 50000 coupez 4 fig. prenez le *cin-*
(*quieme : ainſi des autres.*
Pour diviſer par 2 prenez la *moitié* de la ſomme que
Pour diviſer par 3 prenez le *tiers* { vous voulez
Pour diviſer par 4 prenez le *quart* { diviſer.
Pour diviſer par 5 prenez le *cinquieme*, ainſi pour 6,
(7, 8, 9 & 10.
Pour diviſer par 12 prenez le *tiers* du *quart*,
Pour diviſer par 15 prenez le *tiers* du *cinquieme*,
Pour diviſer par 16 prenez le *quart* du *quart*,
Pour diviſer par 18 prenez le *tiers* du *ſixieme*,
(ainſi des autres, voyez le feuillet 73.

IV. Can we avoid them ?

In many case where we have seen some "etc", "and so on", "...", we could, theorically, express the same idea explicitly (for example, in some cases, by universal quantification, or in writing $\sum_{n=0}^{\infty}$, or else...).

It seems important to notice that information languages do not know "etc"(1).

Whitehead and Russell did not introduce it in their Principia Mathematica (however they use it at the second level in their text).

We could think that we have not to ask to pupils to understand (to guess?) more than to machines and decide to never use this: it is possible theorically and we have seen that mathematiens also prefer that behaviour because all is clearer for them.

But the problem is that didactician know that, often, on the contrary, for example, the sign "$\sum_{n=0}$" is easily read by students only after having many times translated it in a sum of terms followed by "...".

We know that "and so on" is necessary to understand induction.

When Z. Krygowska related ([K1]) an experiment of reading a chapter of Bourbaki, she wrote page 232:

> (C) Problème de formalisation, recherche, discussion: Programmer de la manière la plus simple une échelle sur la base donnée précédemment. Suggestion: il s'avère utile d'écrire le programme en utilisant les nombres qui indiquent, selon des conventions adéquates, les opérations à exécuter et les objets sur lesquels on exécute ces opérations. Les simplifications consécutives des divers projets conduisent finalement à "la loi de la construction" de la suite c_i: l'échelon "produit cartésien" sera programmé par le couple de deux numéros des échelons-facteurs; l'échelon-ensemble de parties d'un ensemble par le couple (n, 0) où n indique le numéro de l'ensemble donné; l'échelon-ensemble de base par le couple (0, n) où n indique le numéro de l'ensemble de base en jeu.
>
> Le compte rendu de cette recherche dans la forme de la définition par récurrence s'avère pour les débutants un exercice assez difficile, mais instructif, un exercice de passage partant de la récurrence intuitive, exprimée naivement par l'expression "et cetera" et se dirigeant vers l'expression formelle.
>
> Le lecteur observé a dû passer par la voie inverse conduisant de cette expression formelle à cet "et cetera" intuitif, ce qui lui a été absolument nécessaire pour pouvoir comprendre. Il n'était qu'un mathématicien d'une expérience scientifique assez limitée, mais il savait se servir au cours de la lecture mathématique d'une technique d'analyse du texte adaptée à ses possibilités at à ses besoins personnels.

The use of "etc" seems to be a necessary approach, but how can we introduce to it?

When H. Freudenthal proposed a language for cosmics intercourses (LINCO [F3]), "etc" was one of the first expressions introduced. He began by:

"1/11 = 0,01010101. Etc.
1/101 = 0,0011001100110011 Etc. "

and a long sequence of other periodic fractions.

He said "this is an important step. A Linco's word meaning "and so on" and written "Etc" is determined. I doubt whether it could be done at an earlier stage if we wish to avoid ambiguous interpretations. The child that computes its first periodic fraction faces infinity. By showing periodic fractions we get the most natural opportunity of introducing "Etc"."

We have no opportunity to verify if the partner would have understood exactly what we wanted but we know that we too have learnt this use through many situations during our childwood. In the examples proposed in "Linco" to "illustrate the use of "Etc"", there are expressions like a^{n+1} = a a^n = a a Etc a a, how can the partner guess that here "Etc" is put for a sequence of (n - 2) signs "a", whereas in the development of $1/_{11}$ it was put for an infinity of "01"? Moreover, we could write " a^{n+1} = a a Etc a " or " b^p = b b Etc b " and you can understand what "Etc" means in each expression, you do but a machine does not; you, men and women, do but perhaps cosmic people do not and above all some pupils do and some others do not.

This linguistic process is interesting because, as we have seen, in some cases it can be simplifying , in others it can be imprecise enough to cause misunderstandings. So, as for many language's uses or abuses, the important point is not to avoid them, but that teachers may be conscious of the difficulties involved in order to control the understanding.

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(1) After this lecture, C. Laborde said that she remarked many times that a great difficulty for her students in computer science, was to write a loop: they write one or 2 or 3 times and want to write "etc"

NOTES:

[A_1]	J. ADDA	L'incompréhension en mathématiques et les malentendus- Educational studies in Mathematics, Vol. 6 Nov. 1975.
[A_2]	J. ADDA	Des étudiants à la loupe - Raisonnement par recurrence en maîtrise de mathématiques - Gazette des Mathématiciens - (Société Mathématique de France) Août 1979.
[B_1]	BARREME	Arthmétique du S^r Barrême - Paris 1788.
[D_1]	D'ALEMBERT	Recherches sur le calcul intégral - Histoire de l'Académie Royale des Sciences et Belles lettres- Année 1746; Berlin.
[F_1]	FISHBEIN	Intuition and mathematical education - Proceedings of the second international conference for the psychology of mathematics education - 1978) Fachbereich Mathematik, Universitat - Osnabruck.
[F_2]	H. FREUDENTHAL	Mathematics as an educational task - REIDEL.
[F_3]	H. FREUDENTHAL	LINCO - North Holland Publishing Company.
[K_1]	Z. KRYGOWSKA	Le texte mathématique dans l'enseignement - Proceedings of the first international congress on mathematical education - LYON 1969 - REIDEL.
[M_1]	MAGNARD Editeur	Mathématique contemporaine - Cahiers - Cours préparatoire - (Thirioux-Gaspari-Roy-Verdier- Paris 1974)
[T_1]	TOUYAROT-HAMEAU	Mathématiques au CM$_2$ - F. Nathan, 1976.

The understanding of place value (numeration)

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General objectives

In addition to the computation skills more or less characterized by automaticity, the understanding of our number system (decimal) involving the idea of place value should be developed in primary school. In recent years, efforts have been made in this direction. However, at the beginning of our research, we felt that "understanding place value" had still a great deal to be clarified and our research project originates from this conviction.

Our aims were four-fold:

1- clarify the notion of place-value.

2- clarify what is meant by "understanding place value".

3- construct a series of problems which could be used.

  a) for diagnostic testing

  b) and as learning units

4- single out the main difficulties and strategies of the children.

Those four aims were reached through the construction of a test* and its administration in interviews.

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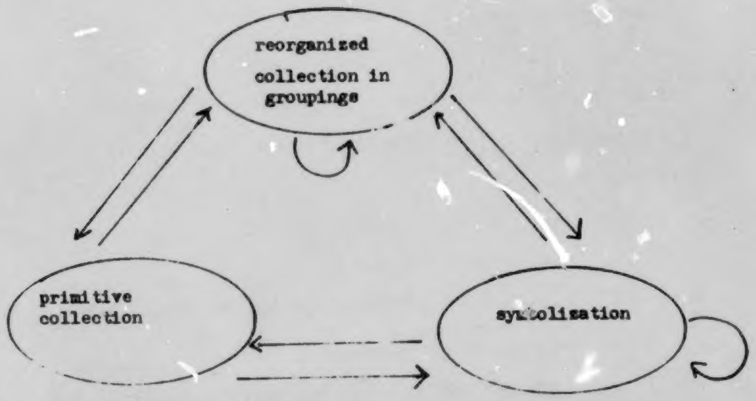
* Which involved as much effort as its administration

Construction of the test

It is important to distinguish "place value" from the idea conveyed by the french word "numération". While "place value" deals almost essentially with the number and the value of each digit, "numération" is more dynamic and suggests the process by which one goes from a collection to a number. So numeration (which we shall use as a neologism from now on) should be understood as a very complex network of abilities which come into play inside a system characterized by three major "states": the collection, the re-organized collection in groupings and the symbolic representation. We worked out a path inside the diagram for each question. As the reader should note, basic abilities such as: to make groupings, to "unmake" them, to go to groupings of higher order, to code, to decode ... are not uniquely characterized by a single path.



In order to categorise the test items and make sure they display enough diversity, we make use of three criteria: the set of abilities and the path in the diagram involved by an item (as shown by the diagram) were classified as easy, difficult and of average difficulty. (Our judgement was mainly based on the intrinsic complexity of the process). Two other criteria were used: Contexts (or situations) were characterized as familiar or non-familiar and the symbolic representations utilized brought in a further categorization whether they were apparent, disguised (non-apparent) or conventional. Categorization was achieved through this system of 3 X 3 X 2 = 18 cells.

We constructed two different tests which together cover the whole 18 cells described above. One was intended for pupils of 1st year (6-7 years old). We shall not examine it in this paper. The other one was designed for pupils of 3rd (8-9 years old) and 4th (9-10 years old) year and comprised 4 questions (11 items). Item 1-b appears in appendix.

	familiar			non familiar		
abilities & path symbolisation	easy	average	diff.	easy	average	diff.
apparent						
disguised						
conventional						

Interviews with the whole test were conducted at three different levels 1st year, 3rd and 4th in three schools, each catering for a substantially different district: a socially deprived one, an average one and an upper middle-class one. In all, about 40 pupils in 1st year, 75 in 3rd year and 50 in 4th year were interviewed by the two researchers and an assistant. In order to get more insights into the process, interviews of groups of 2 or 3 pupils were also organized after the individual ones. They involved two-third of the children but we shall not mention their results in the present paper. The whole testing took place from March to June 1979.

Results

The whole analysis of the results is not yet completed but covers more than what we report here. A final report is expected for next Spring. We shall concentrate in this paper on responses of the 75 pupils of 3rd year on item 1b).

A) Results of 1.b)

A successful strategy (showing understanding) is different from a strategy which yields a right answer, namely: a number bigger than 423.

-We estimate that about 10%[**] of the pupils displayed a good understanding of numeration (place value). The criteria on which we based our judgement will be made explicit after we shall have examined how lack of understanding was operationally defined.

-Of the 90% of the pupils who did not show good understanding, 15% accepted every digit they tossed and jotted them down from right to left or left to right and some even added them up.

-Of the 75% of the pupils left, 40% used a positional strategy which consists of accepting a digit only if it is bigger than the digit of 423 not already covered. In other words, the number was constructed so as to be "bigger in all places".

-20% used combine and/or unsystematic methods.[*] We discovered three additional predominant tactics which sometimes overlapped one another.

-wait for 5 to start with the first cell

-systematically use big digit (4 or 5) even though sometimes the first cell is already filled with a "winning" 5.

-15% impossible to classify (since our interviewing techniques did not allow systematic cross questionning).

-Understanding was interpreted throughout as the complement of lack of understanding. It was among other things operationally defined with sets of actions such as

-select the first 5 coming up and say: "I'm the winner", after having put it in the 1st cell

-accept "0" and "1" and put them in 2nd and 3rd cells when conditions are favourable

-select 4 and put it in the first cell when conditions are favourable.

[**] all percentages are rounded up to the nearest 5%.

[*] intentions of pupils were not always easy to be pinned down.

Partial conclusions

1. Regarding item 1-b. Even though most of the children can compare two three digit numbers as it is asked of them in textbooks, we discovered that the digit-by-digit construction of a number meant to be greater than a given one is a great deal more difficult. The set of abilities or skills built up in class are not transferable to our situation.

2. The mastery of the substraction algorithm involving "borrowing from the next digit", and which is achieved by the majority of the pupils in 3rd and 4th year cannot resist the interference of pictures or cannot be transfered to more concrete actions which involve the idea of borrowing.

3. We observed that pupils of 4th year used computation more systematically than pupils of 3rd year but without being better when they have to resort to more primitive methods and being sometimes launched into a less efficient strategy.

4. The results of the items point out the main difficulties met by the pupil with place value questions: grouping of groupings, the rule of conventional grouping being more complex than the apparent rule or disguised rule.

General Conclusions:

5. As a consequence, our items reveal new facets of what is meant by numeration and understanding numeration (place value).

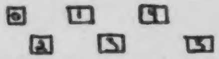
6. A major outcome of our analysis was to observe how similar numeration and measure are. In particular, some items showed that the notion of measure can be used to develop the idea of numeration.

7. Because they cover more thoroughly the notion of numeration, we believe that our problems could be used as very fruitful teaching material and as diagnostic tools to single out difficulties.

Appendix

Problem 1.a)

"You have the following tags:



(Cardboard tags are put on the table)

What is the biggest number which you can make with all those tags"?

-(Two tags are interchanged). "What do you think about my number? Is it bigger than yours?"

-"Can you read this number?"

Problem 1.b)

-"Look I have a number here." (423 is written on a cardboard).

Can you read it?

We're going to play a game. The winner is the first to make a number bigger than this one, well any number provided it's bigger than this one.

Each of us in turn, we'll toss a die. Let's say that 6 doesn't count, well it's zero. You write your result here and then you decide if you take it to make your number. If you take it, you put it there (three blanks are prepared for each player). If not, you cross it out. Don't forget, you must construct a number bigger than this one, any number provided it's bigger".

IMPORTANT COMMENTS

1. The game is played three times.

2. If the strategy is not clear, the interviewer provides a series of digits to the pupil.

3. The strategy used by the interviewer is kept secret. The pupils only see the numbers tossed. They do not see how it is used.

RESEARCH ON TEACHING METHODS IN SECONDARY MATHEMATICS

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At Nottingham we are now in the first year of a modest 3-year research project. This paper describes the lines on which we are working and discusses some of the points arising from the first pilot comparisons of two teaching methods, one concerning directed number, and the other concerning decimals. We are in no sense presenting definitive results here, but we would welcome discussion of the work while it is still at a flexible stage.

The research consists of two separate but related enquiries. The first examines the effectiveness of a set of teaching methods derived from Piagetian theory. The second attempts to evaluate the attainment of general strategies of mathematical activity under different curriculum conditions. In the latter case the development of suitable procedures for evaluation is regarded as a major part of the work. These are related in that the Piagetian concept of learning as an aspect of the child's own development, motivated and directed from within himself, leads naturally to the attempt to teach in such a way that the pupil gradually becomes able consciously to direct his own learning by the employment of general strategies. As members of the Centre have been involved in the development of a curriculum (The South Nottinghamshire Project) based on this philosophy, and this is in operation in a number of local schools, it is possible to test out a number of aspects of the related theory in genuine classroom conditions.

TEACHING METHOD RESEARCH

Existing work at Chelsea College and at Brunel University is attempting to identify key areas of conceptual difficulty in mathematics (Brown 1976, Rees 1974; see also Collis 1975a, b); we are investigating certain fundamental aspects of teaching method which might facilitate learning in these areas. Current theory and practice distinguish two main types of teaching method: exposition and discovery. Exposition may or may not be 'deductive', as when the rules for multiplying decimal numbers including the placing of the decimal point are deduced from the rules for multiplying fractions (as in the Scottish Course Modern Mathematics for Schools). The most common variants of discovery learning are 'guided empirical discovery', as when the rule for the angle sum of a triangle is induced from sets of measurements, and 'guided deductive discovery', as when the same rule is derived by investigating the equal angles in a triangle with a suitable parallel drawn, as in the standard proof. A third class of teaching methods derives from Piagetian work. Dienes' version of this method is the systematic exploration of a concrete embodiment or of a game; examples are his activities with logic blocks, base blocks and the games embodying group structure. Others have used the results of Piagetian studies which reveal what misconceptions children have before achieving understanding of a given concept, and have presented such children with situations which induce cognicive conflict, the resolution of which entails the formation of the concept in question (Smedslund 1961, Fischbein 1975, Sheppard 1973, Inhelder, Sinclair & Bovet 1974).

We are investigating the effectiveness of this third class of teaching methods in comparison with exposition and discovery, in a number of areas of the secondary curriculum. Each study will involve a comparison of one or more methods of the third class with a traditional discovery or expository method, but the particular blend of methods employed for a given topic will depend on the nature of the topic and will be governed by the need for professionally credible methods to be adopted in each case, that is, methods which a competent teacher would recognise as likely to succeed. The aim is to reveal the characteristics of the different teaching methods as well as to judge their overall effectiveness; it is therefore proposed to conduct individual interviews as well as group teaching and testing. The criterion tests are being developed from those in use for the Chelsea research and the results from this and other existing research, including that at Brunel University will be used to indicate the misconceptions on which the 'cognitive conflict' teaching will be based.

PROPOSED TOPICS

This scheme shows which methods will be tested in each topic area, and indicates the main source from which teaching material will be drawn, to be adapted as necessary for the research. The first two are in general first year topics, 3 and 4 are second year, 5 and 6 third year.

1. Multiplication of whole numbers

Objectives: calculation of products (2 x 2 digits)

Methods: concrete embodiment (dot grids) (South Notts Project),
exposition (SMP cards)

2. Multiplication and division of decimal numbers

Objectives: (a) single-digit calculations (e.g. 40 ÷ 0.08)
(b) long multiplication

Methods: for (a) cognitive conflict (e.g. multiplying ⇒ bigger,
0.57 = fifty seven)
for (b) exploration of embodiment (squared paper, South Notts)
deductive exposition (Scottish)

3. Addition and subtraction of negatives

Objectives: single digit calculations

Methods: guided discovery (SMP)
exploration of embodiment (figures in coordinate plane) (South Notts)
exploration of embodiment (familiar non-spatial situations e.g. rise/fall of temperature)

4. Enlargement and Ratio

Objectives: length and area ratios in similar figures; enlargement from a centre

Methods: exposition (Scottish)
empirical discovery (SMP)
exploration of single embodiment (South Notts)
cognitive conflict (e.g. additive strategies for ratio)

5. Combinations of reflections and rotations

Objectives: generalisations about such combinations
2 reflections = rotation or translation,
2 half turns = translation etc.

Methods: exposition (Scottish)
systematic exploration of games (South Notts)

6. Algebra

Objectives: use of letters to express generalisations
formulation of equations from problems

Methods: exposition (Scottish)
exploration of games (South Notts)
cognitive conflict

EVALUATION OF PUPILS' STRATEGY DEVELOPMENT

The first inquiry investigates teaching methods aimed at the learning of particular mathematical ideas. The second is concerned with the learning of general strategies for learning and doing mathematics, for example strategies for making generalisations, for giving explanations and proofs, and for recovering items of skill or knowledge which have been forgotten, bringing in concrete embodiments where these are helpful.

The South Nottinghamshire Project course, as well as making use of some of the exploratory teaching methods which are the subject of the first inquiry, has also a number of course units in which there is no specific content objective, the aim being to develop the strategies of investigation. There are two such units, each occupying about 2½ weeks, in each of the first two secondary years. The specific strategies named in these units are (1) generating examples; (2) making generalisations; (3) giving explanations; (4) extending an investigation; (5) writing accounts of investigations. However, the whole course, not only these units, is meant to be taught so as to encourage the pupils to take over the direction of their own problem solving and their own mathematical learning. A test already used with some SMP classes has scales for the first three of the strategies listed above. This is undergoing development and extension to cover the whole range of strategies for learning and doing mathematics. Both written and interview modes are being used.

A few researches exist bearing on the development of certain strategies. Roughead and Scandura (1968) showed that what is learned through discovery methods is a set of 'higher order rules' or strategies, which are then available for use in solving further similar problems. Lucas (1974) successfully taught Polya's heuristic strategies to college students in the context of a calculus course and thus improved their problem-solving ability, and we have already reported some success in an experiment in teaching proof strategies to sixth formers (Bell 1976). In the two reported attempts to teach strategies to pupils in the 11 - 14 age-range, Wills (1967) obtained successful learning of a strategy for obtaining algebraic generalisations of fairly simple number situations (by moving from small numbers to bigger ones, then to letters), but Post (1968) was unsuccessful in teaching more general problem-solving strategies. The proposed inquiry involves a broader range of mathematical activity, learnt over a longer period and in the setting of a regular school curriculum.

PROVISIONAL CONCLUSIONS FROM THE FIRST PILOT COMPARISONS

Directed Number

The main comparison was between the method of systems and strategies based on our own material and that of guided discovery based on a commonly-used text. In the closest comparison of two parallel classes taught by the same teacher, the former method proved superior both on the skill items in the test and on the test as a whole. Direct observation of the class-room work suggested that only the systems and strategies method contained sufficient practice at performing additions and subtractions for this skill to be at all well established. This method also contained the investigations which took the pupils back to the basic definitions of the operations. By contrast, the guided discovery method tended to assume that a relatively small amount of practice was sufficient and in the later examples most could be done without having to face the more difficult cases of subtraction. The chief differences built into the methods were as follows, insofar that is as they relate to the content learning. There were also differences which would only show up on tests of process attainments.

1. The systems and strategies version included intensive work in the identification of points and line segments with directed number pairs. Rules were then given for adding and subtracting, that is, addition was the combination of displacements on the number line and subtracting was defined as adding the opposite. There was then considerable practice in the use of these definitions, aimed at developing fluency in the use of the two rules. The remainder of the unit consisted of applications and investigations in the four quadrant co-ordinate plane involving the finding of, for example, the fourth point of a parallelogram by adding and subtracting to obtain line segments and to add the number - pairs for these to the number - pairs for the appropriate points. There were also several pure mathematical investigations, in which the commutative and associate laws and the relation between addition and subtraction, that is that $a - b + b = a$, were proposed as questions for investigation, first by considering a number of particular examples but then by the provision, if possible, of a general argument. This mode of approach to the investigations was the subject of a taught lesson.
2. By contrast, the guided discovery method consisted of shorter sequences of more varied questions ranging over a somewhat greater variety of situations. Thus there was a difference in the level of mastery sought and achieved on the operations of addition and subtraction. Secondly there was a difference in approach to the giving of reasons for the definitions of addition and subtraction. The guided discovery method attempted to make the definitions plausible by first introducing them as newly invented operations and subsequently showing empirically that the same results were obtained with these as with the ordinary plus and minus in counting numbers. The systems and strategies method made no such attempt and it could be said that such work was replaced by the attempt to embed the new operations in the old in the course of the pure mathematical investigations. These occurred after the skills of addition and subtraction had been well learnt.

Decimals

In this case there was no significant difference on the tests overall. The two methods, that of the concrete embodiment i.e. squares on graph paper as the embodiment for decimals in tenths and hundreths, and rules i.e. the giving of rules about counting numbers after decimal points to place the decimal point in the product, and for multiplication by integer, of keeping the numbers in the right columns. It was clear from the observation of the teaching that for the achievement of skill in computing products with decimals, the method of direct teaching with rules was superior and that in practice the method of the concrete embodiment, though contributing to the understanding of the relation between fractions and decimals e.g. 90 hundreths = 9/10s = .9, was not an effective support for the rules of multiplication themselves, since the chain of implicit argument was too long for these pupils to appreciate (this was performed with two below-average third year classes, both with the same teacher).

TENTATIVE CONCLUSIONS AND POSSIBLE NEXT STEPS

1. Mastery. On the face of it the practising of a skill up to some level of mastery which carries confidence on the part of the pupil would seem to be merely a sensible teaching approach independent of any particular theory. However, we see it appearing in the fairly stringent criteria for success adopted in Piagetian research when a concept is not considered to have been acquired until it can be explained and can be transferred to any new situation. There is also certainly evidence for the weak retention of ideas and skills which are not well integrated into cognitive structure. Thus it would seem important on practical grounds to pursue this point, that is, to see whether the use of the mastery criterin in other topics was as successful as in this one and also that this could sensibly be related to theory.

2. Deductive explanations. The theory that pupils do not think of mathematical ideas as constituting a deductive framework or that operations on directed numbers are entities which are open to invention comes directly from Piagetian theory and was also evident in my own research on explanation and proof. The superiority of the intensive exploration of an embodiment of the system of relationships to be learnt as against such deductive explanations seems worth pursuing also in other topics.

3. Out of the decimal study comes the question of the inter-relationship of different kinds of objective, computational ability and the understanding of the meaning of decimals and of their laws of combination and also the ability to recognize situations to which the operations of multiplication, division etc. apply. These appear to be quite independent acquisitions. But are they? The key question here seems to be, what kind of experience can help the identification of operations in a practical situation. A study of this question would incidentally illuminate the problem of what calculation is needed in the presence of calculating machines.

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SCAN - a systematic classroom analysis notation

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In order to judge the effect of any new resource, be it a work card, film, book or computer program, we want a detailed picture of the pattern of teaching and learning in lessons both with and without the resource. We have designed the SCAN system to provide this information, initially in the context of mathematics teaching. It provides a notation for describing a lesson, as it takes place, in terms of natural units of interaction on three different levels of time scale - the "remark", the "episode", and the "activity". Remarks of various kinds (questions, explanations, etc.) are qualified by descriptors indicating depth and level-of-guidance. Additonal information on the teacher's strategy, the content, and the resources used is added together with copies of pupil's work. When a computer program is among the resources used a complementary analysis of the style of its design and use is essential; we have identified a number of dimensions for this classification, some relating to the program structure and content and others to the way in which the teacher chooses to use it. The aim of these observations is both to enable the creator of the program to construct better programs, and the user to incorporate them into his teacher more effectively; other applications, in teacher training for example, are evident.

The work is itself still in the process of development and evaluation. Impressions so far are informal. However, good inter-observer consistency is achieved with little training. Record and readback by different observers on a video lesson has produced a graphically realistic description of the teaching style employed, a style which was separately observed to be substantially modified when the same teacher was teaching using a program teaching unit, developed by the Investigations on Teaching with Microcomputers as an Aid project. The records give specific information which tells far more than an unstructured impression of the lesson by an experienced observer; as well as providing quantitive information on the style of teaching, it can give a qualitatively different impression of the lesson. Thus we have found it a useful tool both in curriculum development and teacher training. We are now verifying more carefully these and other hopeful features of SCAN, as well as developing it further.

Here we have space to give only a flavour of what a SCAN record looks like and how it is compiled; a full write-up is in course of preparation. In order properly to appreciate the notation, it is best to use one of the training video tapes that are available at the Shell Centre. The notation is summarised in the key on the next page. The following is an example of a SCAN record of a small section of a lesson in which the class is working idividually (W1) with work cards (PMC); the dialogue (D) between the teacher and individual pupils is recorded as a series of remarks (separated by /) comprising episodes (//), the remark / qα1✓ / on the first line below, for example, denotes a teacher question (q) of simple recall (α) where the teacher's guidance is such that the pupil recognises there are only a few alternative possibilities (1) and answers correctly(✓).

Resources Used	Activity	Remarks/Episode Summaries
PMC	W1 D	o|cf|ch|iα2||CO||pch|eβ1|qα1✓|eα1||CO
		pch|cf|iα1|iα1|e|pa|cf||co|| pq|eα1||CO
		pq|e|| cn || i||co|| ch|qβ2o|qβ1✓
		qβ2o|∧✓||co|| o|cf||CH|| i||I||m||F

The dialogue described in the first line was the following; the classification of each remark and episode is given in the right hand column. (Of course, a much fuller justification of these classifications is demanded - in this brief note, however, we must just ask you to accept them.)

T:	// Yes, that's nice / but what does it mean? What is it?	|o| |cf|
P1:	I plot this against this./	|ch|
T:	What are you going to write? A title. The most important thing about any sort of chart is a title.//	|iα2||CO||
P2:	Do I have to write that down like that?/	|pch|
T:	You've already got it like that in base 5. Now you've got to write it down as an ordinary number,/ so how may stars have you got?	|eβ1|
P2:	I've got 13./	|qα1✓|
T:	So that's what you write down. That's in base 10.//	|eα1||CO||

SCAN CODE SHEET (SIMPLIFIED VERSION)

SCAN ON

ESOURCES	ACTIVITY LEVEL	EPISODE LEVEL	LINGUISTIC DESCRIPTORS	REMARK LEVEL QUALIFIERS
M - teacher produced material	E - Exposition, teacher to group of 5 or more	D - Defining	g - gambit	Prefix p for pupils, except in PP dialogue when prefix t for teacher remark.
MB - printed material (books)	D - Dialogue, teacher to group 5	I - Initiating activity	m - managerial	**Depth of demand on the Pupil**
MB - printed material (cards)	W$_n$ - Pupil work in groups of n	CO - Coaching	ch - checking question	α - recall, single fact, single act
- computer	PP - pupil-pupil dialogue	E - Explaining (new material)	q - question	β - exercise, straight sorward, several acts
HP		R - Revising	e - explanation, includes definitions	γ - extension, new ideas
S etc		SS - Searching (successful)	a - assertion	**Situation or level of guidance**
		SU - Searching (unsuccessful)	x - giving example	1 - highly stuctured, close direction, small number of choices
		F - Facilitating	i - instruction, initation	2 - some guidance, requires connection rather than selection
			cf - confirmation	3 - minimum guidance, open
			r - rejection	
			k - correction	
			o - observation	
			∧ - repeat remark	
			FATE OF QUESTIONS	
			✓ - correctly answered	
			x - answer incorrect	
			o - no pupil response	

In practice, observers have little difficulty with the linguistic descriptors but, not surprisingly, tend to diverge rather more in assigning qualifiers giving the depth of demand on the pupil and the level of guidance provided by the teacher; these depend on the context, and on the pupil's previous experience. However, after an initial training session of only an hour or two, typical observers produce similar records which show clearly the characteristic rhythms of the teachers dialogue and style. Further notes on the content being covered, the degree of pupil involvement, and detailed teacher objectives are added to the SCAN record, which also include a note of the actual words of any γ- question (one which seeks to extend the students previous experience).

The system has been developed on video recordings of mathematics lessons, without and with a microcomputer as a teaching aid; this is, of course, essential if detailed discussion of remark classifications is to be possible. However, we have found it possible, indeed easier, to produce SCAN records in real time in actual mathematics lessons. Indeed we have regarded this as a necessary requirement of a system that is to be sufficiently economical in use to be of value in curriculum development and teacher training, rather than in educational research. For these last purposes, full recordings and transcripts of the lesson, bringing much additional information, will often be of value, though even here recording at the SCAN level of detail has a lot to contribute.

SOME REFLEXIONS ON
THE CONSTRUCTION OF THE IDEA OF NUMBER IN THE SIX - YEAR - OLD

by : Annie BESSOT, Claude COMITI, Claude PARISELLE

At the 1st conference of the I.G.P.M.E. (1977, Utrecht), we presented a study on the six - year - old's approach to the natural numbers. This study was concerned with the analysis of procedures used by children in the Cours Preparatoire (first year of compulsory education in France, 6-7 year old) in the region of the number 15 in two situations : construction of a collection of objects with the same number of elements as a given collection, and construction of a collection with more elements than a given collection.

The conclusions of this study led us to continue our research by clinical interviews* with children who had just entered the Cours Preparatoire. The object of these interviews was to entail, to observe and to analyse certain behaviour patterns relative to the use by the child of the "comptine numerique" (chanting or repeating of sequences of numbers such as " one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen) which he has in his memory at that time.

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To illustrate these interviews, we have chosen to give here certain behaviour patterns of children in situations that we have proposed to them.

When we ask (Ia)*:
"Do you know how to count ? Up to where ? Show me", most children know how to recite the comptine correctly up to seven or eight, some get to sixteen and some, very few up to thirty.

But when we ask (Ib)*:
"How many matches are there on the table ?"

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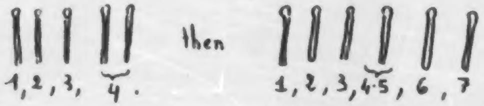
*cf. Annex

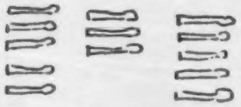
Sometimes, there is a lack of adaptation of the word (one, two,...) to the action accompagning the counting, as if the fact of touching an object with the finger and that of saying a number were two acts which the child makes at the same time without feeling the necessity of one-to-one correspondance.

Marion proceeded as follows :



When one increases the numerical range, the child only counts efficiently and is sure of himself when he has experienced the need to organise himself before counting or in the course of counting.

Frédéric knew how to recite the "comptine" up to fifteen. He counted seven matches without problem. When he was presented with thirteen, he found twelve and only finished after having arranged the matches in the following way :



And when we ask (Ic)* :
"And now ? How many are there ?",we see that a lot of children, at the beginning of C. P. can well associate 1 with one match, 2 with another,... 7 with the last match counted, without however being sure to obtain the same result after modification of the position of the matches.

Even if the child possesses this certainty for seven matches, it is possible that he no longer has it when one passes to a larger numerical range.

Frédéric, for example, who managed the preservation of cardinality for seven matches experienced afterwards the necessity to recount the thirteen matches which he had just counted when their place was changed. He found there were fifteen and was not surprised !

In our later research , we found that this problem is not resolved by the end of C. P. when one increases the numerical range, for a great number of children (about the third part).

In (IIa)* "Now you are going to put N matches on the table", children experience more difficulty in extracting from a collection a collection of a given cardinality than in numbering a given collection.

Rose-Marie who was asked to place sixteen matches on the table counted all the matches in the box from one to twenty-two and said : "there are not enough". After she started again, she declared : "it is not possible, there aren't sufficient" although she had just said aloud : "14, 15, 16,....17, 18, 19, 20, 21, 22" while counting the counting of the box.

In fact, she didn't have a sufficient command of the number sequence to place the number sixteen correctly in the sequence. So she couldn't anticipate it and she allowed it to pass. Hence it would be a profond error to believe that a child who can recite perfectly the number "comptine" in a certain numerical range appreciates the ordinal aspect of number. In reality, he only knows the "comptine" as a "song" without perceiving the numerical relations which structure it.

In (IIb)* and (IIc)*, when a match is added or taken away, some children recount all the matches, others reply without recounting.

For these last children, the reponses are as spontaneous for the predecessor as for the number which follows, when N doesn't go beyond eight. Beyond that, even if they find the following number, they have to recite to themselves again the "comptine" in order to refind the predecessor.

"I counted in my head up to fourteen because the next was fifteen" said Paula ; Michaël, on the other hand, gave eleven as the predecessor of thirteen"because I counted 1, 2, 3,....., 10, 11, 13."

Here one sees that the notion of predecessor and following number can function very well without a perfect knowledge of the "comptine".

In our later research we found that this problem is not resolved by the end of C.P. for all children.

This problem continues until the end of C.P. : when children are asked to find the predecessor of thirty
- some know how to use the rules of numeration in order to give immediately the reply : twenty nine
- others recite to themselves a part of the "comptine" (for example : "25, 26, 27, 28, 29, 30. It's 29").
- but there are other children who do not know how to do otherwise then recite it from one.

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* Cf. Annex

In part III, in order to determine the numerical relation which exists between two given collections, a larger number of children rely upon their perception and don't experience in any way the need to have an objective method of being sure of what they claim. Their intimate conviction is sufficient for them.

However, when the experimenter asks them "How do you know ?" or "Are you sure ?", certain children feel then the need to justify their reply by the use of counting or an element-to-element correspondance.

When we ask Frederic (IIIa)* :
"Show me which side has more objects ? Are you sure ?", he replied immediately "There are more matches." When he was asked how he knew, he replied : "because I have seen it : here there are a lot, there, there are less".

But when we ask him (IIIc)* :
"Are there more pieces of cardboard or more macaronis ?", he replied : "there are more pieces of cardboard because there are some on top of the others". And when he was asked : "are you sure ?" he began to make a one-to-one correspondance and said : "there are the same number in each, there are none lacking, there are not too many".

In (IIIb)* when children are asked to transform two collections of inequal size into two collections of equal size, most of them encounter difficulties : some of them manage only by successive approximations relying sometimes on a spatial organization.
Thus Michael put his counters in line, took a counter, recounted, took another counter, recounted.
Frederic, in the other hand, declared : "it's necessary to put more counters". And, while seeking an approving regard, added one then two then three up to fifteen counters.

One can ask oneself whether the problems found in (IIIb) do not come from the fact that the child finds himself confronted with several tasks in succession :
- to recognize what it is necessary to do to make the collections have the same size (that is, to add counters)
- to find in addition exactly how many objects it is necessary to add in order to obtain the relation desired (as much as) and not another.

On the other hand, when in (IIId)* the children have to transform two collections of equal size into two collections one of which has more elements than the other, they propose immediately to add macaronis and add generally a handfull (very rarely only one).

Finally when we wanted to study in (IIIe)* whether the child claims : " B has more elements than A", when the number of elements of B is diminished one by one (so as to obtain at the end a set of the same size as A) we observed the following behaviour patterns :
- some children confuse the comparison between B and A and the comparison between the two successive states of B
- some children pass directly from the relation "more than" to the relation "less than" without feeling the necessity of passing by "equal to".

Thus Patrick added two macaronis. When one was removed, he said : "there are still more macaronis". When another was removed, he said : "there are less macaronis".

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These interviews have permitted the emergence of certain questions :
Confronted with problems of enumeration or comparison of two collections of objects :

- when, why, and how does the child make the passage from a conviction based upon perception to experiencing the need for an objective criterium ?

- how is constructed the need for an organization without which the child can't reply with certainty to questions relative to these problems ?

- what are the different steps for getting working a method ?

- what is the place of counting among other possible methods ? How is the place related to the type of learning at school ?

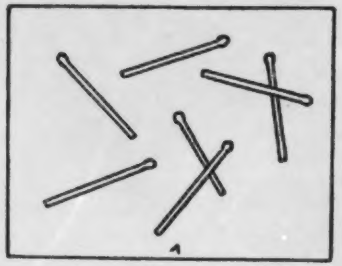
- how does the pupil develop a knowledge of the relations which exist between the numbers considered as elements of the numerical sequence. How does this knowledge evolve ?

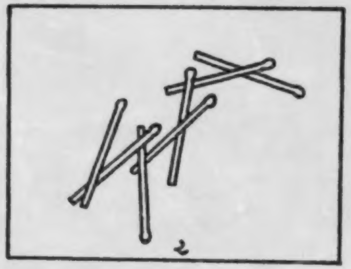
ANNEX

CLINICAL INTERVIEW

I. The first question asked, at the same time destined to give confidence to the child and to give us an idea of the extent of his numerical knowledge can be decomposed into three steps :

a) do you know how to count ? Up to where ? Show me.
b) how many matches are there on the table ?
c) and now ? How many are there ? (after having pushed the matches, before the eyes of the child, from one end of the table to the other).





II. The second series of tasks had for aim the construction by the child of a collection of a given cardinality and the verification of the working of the notions of preceding and following. It had four steps :

a) now, you are going to put N matches on the table.*
b) you see, I put another one there. Tell me how many there are now.
c) make it so that there are exactly N matches on the table.
d) I remove one. How many are there now ?

* N is a number which the researcher chooses in the range of numbers where the child can easily recite the "comptine".

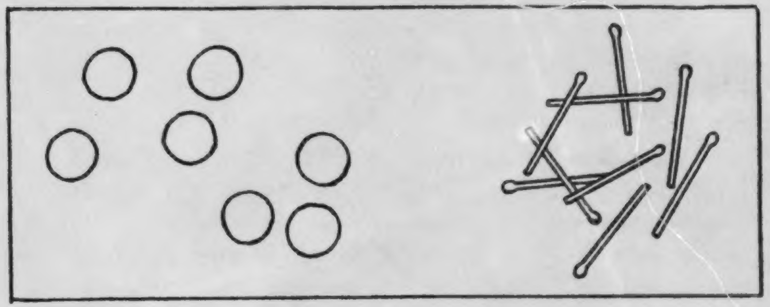
III. The rest of the interview had two purposes :

- to see if, in a situation imposed by the researcher, the child is capable of determining the relation existing between two collections present (of equal size or not) and to study how he goes about determining this relation.

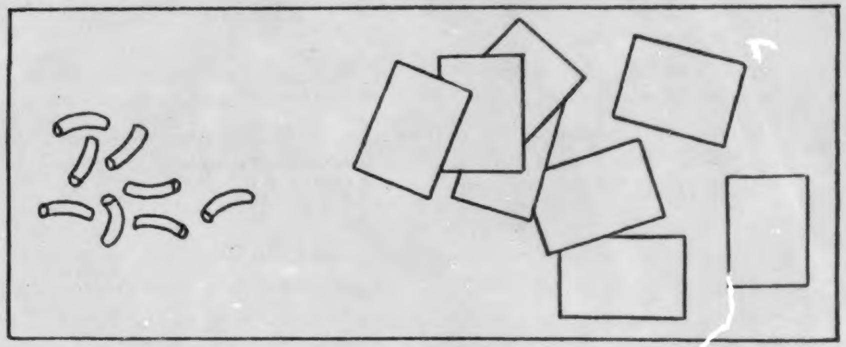
- to see if, having at his disposition two homogeneous collections A and B, as well as two boxes in which there are available other elements identical to those of A and B, the child is capable of managing to transform A and B into two collections which verify a certain relation imposed.

- finally, to study the preservation of the relation "B has more elements than A" by reducing one by one the number of elements of B until the set obtained is equivalent to A.

a) show me which side has more objects. Are you sure ?



b) what can you do so that there is an equal number of matches and discs ?
c) in your opinion, are there more pieces of cardboard or more macaronis ?



d) how could you arrange it so that there are more macaronis than pieces of cardboard ?
e) and now, are there still more macaronis ?

Note : naturally, the children can operate on the objects which they can touch and move (but not on the drawings).

Visual abilities and mathematics learning

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1. Introduction

In this paper I intend to explore the interface between two sets of psychological constructs. One set concerns the visual and spatial field and the other, the learning of mathematics. But rather than do a conceptual analysis or report an empirical investigation, I have chosen to review the different research emphases which have contributed to our knowledge in this area, and to examine what they can, or cannot offer the mathematics educator. I hope to show that this literature does contain many potentially fruitful ideas and approaches which we in mathematics education can use either indirectly, in the classroom, or directly, to guide our own research efforts. Also it will be seen that the goals of research psychologists may lead them in a direction which is away from our concerns, and that therefore we must exercise caution and keen judgement in selecting that which will enable us to further our own developments.

2. Factor analysis

The review begins with one of the main lines of research in the area - the work of the factor analysts. Historically, spatial abilities have been of interest ever since Galton (1883) began his systematic psychological inquiry. When, later on, the supposedly more objective methods of the factor analysts held supremacy over those of the introspectionists, researchers like Spearman (1927) and Thurstone (1938) attempted, by their increasingly more sophisticated statistical methods, to clarify in their various ways the structure of human intelligence.

Spatial ability and numerical ability were usually represented in all the large factor analytic studies but mathematical ability as such, was not usually their concern. Many researchers appeared to lean towards the idea that mathematical ability was a second-order ability, dependent on a certain combination of primary abilities. Thurstone, with his focus on Primary Mental Abilities, was the main proponent of this viewpoint but others appeared to share the same belief.

In the forties and fifties mathematical ability, or 'the mathematical factor' became the focus of more studies and researchers such as Barakat (1951), Werdelin (1958) and Wrigley (1958) all wrestled with the problem of clarifying the nature of mathematical ability.

Unfortunately their findings are unclear and certainly inconclusive. The relationships between spatial abilities and the other abilities (including mathematical) differ from one study to another. MacFarlane Smith (1964) in his magnificent book on spatial ability does a detailed analysis of these studies in his chapter concerning mathematical ability, but his case for arguing that spatial ability is the key ability underlying mathematical ability has still to be proved. For example in Werdelin's massive work, there were no significant loadings by the mathematics tests on the spatial factor.

But it is too easy to get drawn into the factor analysts arguments over the uniqueness of the mathematical factor, or the correct combination of component abilities, or the methodological issues. The factor analysts, as their name suggests, are principally concerned with analysing intellectual factors and their interrelationships. They are part of the psychometric tradition which is based on objective assessment and the quantification of intellectual abilities. They make no reference to individuals, they rarely pursue how an individual approached a solution to a particular problem, and they are rarely concerned with the classroom. Their legacy can be felt to be a disappointing one for mathematical educators, as Krutetskii (1976) clearly argues.

However, I maintain that there is a great deal which has been, and can still be, gleaned from their work. For example they produce many tests, and their analyses clarify many different types of intellectual task. The work of Guilford (1967) and his followers, for example, is in the true factor analysis tradition and has given us many ideas, e.g. creativity (fluency, flexibility). In the area under discussion here, there are many spatial tests and types of spatial reasoning items which we can use in both testing and teaching mathematics. Also despite the lack of clarity of the 'mathematical factor' many correlations appear in the literature between spatial tests and the various mathematical tests. Those between spatial and geometric tests appear to be higher than those between spatial and numerical tests and when one sees the similarities between transformation geometry and some spatial tests one must expect even higher correlations. However, as will be seen later, one must not only think of spatial abilities in connection with geometry. That this tradition is not dead, nor irrelevant, can be seen by the recently published article by Guay and McDaniel (1977) who considered mathematical and spatial abilities in the sex-difference context.

Furthermore, the ideas concerning the appropriate combinations of primary abilities which make up mathematical ability can be very important when considering the teaching of mathematics. It could well be that mathematical ability can best be developed not by just teaching *mathematics* but also by suitably emphasising those primary abilities on which mathematical ability leans. This would focus our attention more on the abilities necessary for doing mathematics, and away from the particular curricula content of mathematics.

3. Developmental psychology

That last point suggests one way of approaching the development of mathematical abilities in children, and it is a point not often made about the work of the factor analysts. It has certainly escaped the attention of those who have seized upon the work of the developmental psychologists as being the only useful psychological tradition for the purposes of mathematical educators.

The massive work of Piaget and his followers is of course at the heart of this research tradition see, for example, Piaget and Inhelder (1967) and Piaget, Inhelder and Szeminska (1960). The influence of the Geneva school is still enormous despite experimental evidence which disputes various findings, despite arguments over ages and stages, despite detailed criticisms over the language used in the research (particularly mathematical) and despite vast methodological debates. Two recent publications on spatial and geometric concepts from the Georgia Center, Martin (1976) and Lesh (1978), report research which was carried out almost entirely within the Piagetian tradition, with no reference being made to any of the factor analysts' work.

However, the Geneva school is not the only source for ideas regarding the development of spatial abilities in mathematics. The work of Werner (1964) is not as well known as it should be, though geography educators appear to recognise the importance of his notions of spatial development (see Hart and Moore, 1973). His *orthogenetic principle* states that: "insofar as development occurs in a process under consideration, there is a progression from a state of relative globality and lack of differentiation to a state of increasing differentiation, articulation, and hierarchic integration". Werner's ideas clearly differ from Piaget's, though there are some similarities.

In addition, Werner does not get into difficulties by using Piaget's 'mathematical' terms, which have to some measure both enabled mathematical educators to see a relevance for their subject, and also have obscured some of the real issues in Piaget's theory. Furthermore, one does not get the impression from Werner's work that he was trying to fit the data to his theory, nor that he was blinkered in his observations by his theory. Werner's work reminded *me* that insofar as we are concerned with spatial ideas in mathematics we must attend to large, full-size space, as well as to space as it is represented in models, toys, and drawings on pieces of paper. It is no coincidence that geography educators find his work of interest.

Another view of psychological development is that of Bruner (1964) whose levels of representation "enactive, ikonic, symbolic" bear some relationship both to Piaget's stages and also to Werner's three levels "sensorimotor, perceptual and contemplative". Interestingly Bruner seemed to develop these ideas in the context of algebraic mathematics rather than within geometry. Also it occurs to me that were his first two levels to be fully exploited, there would be a similarity between this approach and the idea I took from the factor analysts' work of strengthening the 'component' abilities on which mathematical ability depends.

In contrast with the factor analysts before, there is a clearly discernible link between the interests of the developmental psychologist and the interests of the mathematical educator. However, there are also differences, and it is very easy to be drawn into debates and arguments regarding fine points of developmental theory. It must be remembered that a developmental psychologist is essentially interested in the 'natural' development of the child, whereas an educator, because of his interventionist role, is essentially interested in 'unnatural' development.

Therefore, although one can be guided by developmental psychologist descriptions, the task still remains of producing curricula sequences and teaching approaches which will achieve the goals of mathematical education. The work of van Hiele (1959) is an excellent example of what can be done, and I find it interesting that his idea seems to have more in common with Werner's than with either Piaget's or Bruner's.

4. Individual differences

Of course the research methods of the two previous groups are very different, with the "clinical interview" being at the heart of the developmental psychologist's procedures. This is another reason why educators feel some empathy with this school, one works with individual children. However there remains an essential disagreement - a developmental psychologist interprets the main differences between the children he interviews in terms of a different stage or level of development. An educator does not necessarily interpret the differences that way - indeed within any one class of children of a similar age, a teacher may not note *any* difference of level, but he will notice other differences.

A researcher working with children of the same age on a specified set of problems will also notice individual differences of procedure, and this research falls into the "Individual Difference" tradition which is also a long one, see Anastasi (1958). Within the more specifically mathematics literature, the main differences which have been considered are the male/female differences, the gifted/less able differences, the culture differences, the cognitive style difference, and particularly relevant for our purposes here is that difference described by Krutetskii (1976) as the "analytic/geometric" difference.

This last individual difference - the extent to which an individual was more, or less, visual ideas in solving mathematical problems - is one of the most frequently occurring differences in the mathematical education literature. The data range from accounts of problem solving by eminent mathematicians, through various research reports, to teachers' descriptions of their own children. Krutetskii is the latest, in a long line of researchers, to document that difference. Typically, in research regarding individual differences the documentation and description of a particular difference is the first stage. The next point is where, once again, we notice that the psychologist and the educator pursue different goals. The psychologist is usually concerned with why the differences exist, while the educator worries more about what should be done about them, if anything.

So there are many psychological studies which have involved within-individual reasons for individual differences, such as the person's heredity, their genetic composition, the lateral dominance in their brain, and hormonal and neurological reasons. At the present time mathematics educators do not find a great deal there to interest them, though Wheatley and others (1978) have made some intriguing speculations concerning hemisphere specialisation and cognitive development.

It is when there is consideration of outside-individual variables that the research moves more towards the educator's interest. Cultural determinants, social background and societal role-precriptions are all in the educator's domain, and could potentially interest the mathematics educator. Differences in spatial ability and the boy/girl differences have been extensively studied with reference to these variables, see e.g. Mitchelmore (1976) and Fennema and Sherman (1978), and it appears to me that interest in these more sociological variables is growing.

5 Teaching Experiments

Meanwhile, what does the mathematics educator do about individual differences? The research suggests that we go in two different directions, one concerned with the possibilities of remediating those who are 'deficient', and the other concerned with how best to arrange the teaching so that those who have some particular 'superiority' can use it to maximum effect. They are both concerned with what it is 'possible' to achieve, but they are based on very different assumptions about what is desirable to achieve.

The first lies clearly within the 'remediation' tradition and explores the extent to which spatial and visual skills can be taught, or trained. This research varies from studies where the training takes the form of sophisticated coaching on problems very like the test items, to controlled teaching experiments. Furthermore, it is important to remember that such methods which are devised have a potentially wider use than just with the children who are considered to be deficient at a given time. Saunderson's (1973) research shows what can be achieved on spatial tests, and Vladimirskii's (1971) work uses a different approach in the context of geometry teaching. Such findings will be of no surprise to teachers in the technical education field but it is my experience that the mathematics education community does not fully recognise the desirability or the possibility of such training.

The other approach to the 'problem' of individual differences is embodied in what is referred to as the ATI approach - aptitude treatment interaction. Two recent examples by Webb and Carry (1975) and by Young and Becker (1979) show how one can experiment with different teaching methods designed to exploit the different strengths and preferences of individuals. Appealing though this type of experimental approach may be, it is fraught with difficulties concerning the design and control of the appropriate methods. Also the quality of the ideas which have emanated from these experiments are not yet commensurate with the quantity of work involved in carrying them out. However, the definition and clarification of teaching methods which are being carried out should benefit the mathematics educator in the long term.

Conclusions

While discussing these four main research approaches to the mathematical/spatial interface I have been conscious of omitting various studies which don't really fall uniquely into any of my categories. Three of the more important of these are Shepard and Metzler's (1971) work on the mental rotation of three dimensional objects, recent work on the logical classification of spatial tasks (see Clements 1978), and work on the language of spatial ideas, well documented by Olson (1975). All of these can enable us to make significant advances in our field.

In conclusion therefore, I have tried to show that there are many different approaches to research on the mathematical/spatial interface and consequently there are many different definitions, theories, tests and ideas for methods and materials. It is a rich field for mathematical educators to study provided that they are constantly aware of the dangers of becoming embedded in one psychological tradition.

So, although rival schools of research can argue about who's theory is right, the mathematics educator knows that nobody can be right. Our aim is not the development and validation of any one theory, but rather the exploration of fertile ideas which could help more children realize the potential for mathematical activity that they all have.

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TEACHER TRANSFORMATIONS

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Various recent studies in the U.K. have shown that a substantial proportion of the student teachers training for primary schools dislike mathematics and are not very good at it [1,2,3]. It is essential that this cycle -- of poor mathematics teachers passing on their fear of the subject to pupils who may themselves become teachers -- is broken through teacher training. Two problems exist: first, the difficulty of changing a student's attitude to mathematics especially after a long history of failure, and second diagnosing and remodying the important gaps in the student's understanding.

Our approach to these problems is to make extensive use of the activity of computer programming. Now asking such students to program a computer poses problems whose difficulty depends crucially on the kind of computer facilities available and the kind of tasks that the students are asked to undertake. Previous attempts to evaluate the effects of programming on the understanding of mathematical topics and on attitude to mathematics give promising but not clear-cut results [4,5,6,7,8]. We make substantial use of drawing devices connected to the computer, which is programmed in the language LOGO [1]. This allows us to illustrate various important matheuatical ideas visually and to avoid the rather dull programs often associated with programming in mathematics at this level, such as computing the mean of a set of numbers or working out one's income tax.

A small pilot-study was conducted [9] using volunteer student primary teachers from a local College of Education. The objectives were to discover the students' mathematical difficulties, to chart their progress in learning LOGO, and to observe the effects of the programming work on their mathematics using case study techniques.

Mathematical Difficulties

The students took Rees' test [10] and incorrectly answered much the same "Common core" of difficult questions as her subjects had done. These questions included sums involving fractions, decimals, squares and square-roots. But observation of the students while teaching, and interviews with them, revealed many mathematical difficulties not detectable in this kind of test. These difficulties had consequences just as serious for their teaching and were concerned with the students' understanding of, and ability to describe, basic mathematical ideas, rather than their ability to do calculations or solve mathematical problems. The students were often unable to justify and explain the mathematical rules that they taught and knew how to apply. They also had difficulty expressing mathematical ideas clearly in English.

The central difficulty was that the students had learned parts of their own mathematics "instrumentally" and this severely limited their ability to teach their own pupils "relationally" [11]. Many of the observed difficulties are commonplace: why does the rule about dividing fractions work? Why does one add decimal places when multiplying decimal numbers? Why are there two scales on a protractor, and how does one decide which one to use? Some of the unclear expression of mathematical ideas were more unusual. For example, one student persistently substituted phrases of the type 'ten from eight' for 'eight from ten' in her classroom questioning. She explained her mistake as a literal translation from mathematical notation to English. That is, '10 - 8' became 'Ten from eight'. In another case a student teacher mixed up the sharing and partitioning applications of division. She started with an explanation based on partitioning and then continued with phrases and activities based on sharing, and summed up finally in terms of partitioning.

The student teachers knew that they had difficulties with mathematics (that is why they had volunteered for the study), although they were often uncertain about their nature, as in the examples above. Awareness of their lack of ability sapped their self-confidence so that some of them doubted their grasp of topics which they clearly did understand. This lack of confidence caused one student to over-prepare her lessons to the extent that an unexpected event completely disrupted her plans and left her unable to continue sensibly. Another student was convinced that she could not judge distances, and presented this as firm evidence that she was mathematically incompetent. In fact, she was able to judge distances perfectly well by reference to the known dimensions of her own sitting-room, which she could visualise. But she felt that this personal yardstick was too idiosyncratic and that proper mathematicians had some other method of doing the task. It was not her competence at judging distance that worried her, but her use of a personal method unrelated to textbook mathematics.

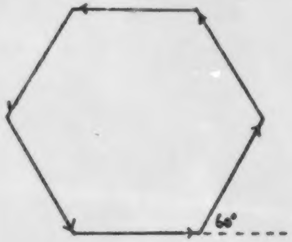
Very often mathematics lessons were a trial in that students realised the disparity between their belief in teaching for understanding and their ability so to do. Some also dreaded being found out by a cild's question or difficulty, as the following quotation from a student illustrates:-

> "Usually I'm absolutely terrified whenever any of the children in the school asked me about angles. Because I would just have to go away and really think about it. If they want an answer right there and then, I usually had to fob them off with something else."

Programming

The students undertook programming projects determined by their mathematical difficulties. Their main purpose was to use the programming language and the computer's ability to draw pictures to provide visual illustrations of basic mathematical ideas. The most appealing part of LOGO for the students was Papert's Turtle Geometry [17]. This is the study of that class of plane figures produced by a small computer controlled, motorised cart with an attached pen. The cart, called a Turtle because of its shape, responds to commands either to move FORWARD or BACKWARD in the direction it is currently facing, or to rotate LEFTwards or RIGHTwards on the spot. The figure shows a program to draw a hexagon. Notice that the rotations correspond to the exterior angles of the hexagon.

```
DEFINE "HEXAGON
FORWARD 100
LEFT 60
FORWARD 100
LEFT 60
FORWARD 100
LEFT 60
FORWARD 100
LEFT 60
FORWARD 100
LEFT 60
FORWARD 100
LEFT 60
END
```



Even such simple programs were not always written correctly the first time, and the computer often produced pictures unintended by the students. Success in completing a drawing gave the students a great deal of pleasure. One student became intrigued by drawing such polygons and was able to use the activity to derive and explain various angle properties of regular polygons, such as the relation between the number of vertices and the total interior angle. She extended her analysis to star polygons, which were drawn by programs similar to that for the hexagon. The student continued by considering the symmetry properties of regular polygons and was able to make what for her were genuine mathematical discoveries, such as the relation between the exterior angle of the polygon and its rotational symmetry. Here the computer functioned as a laboratory where her hypotheses were instantiated as simple programs and then tested. This was very enjoyable for the student and it gave her a strong sense of having achieved something of personal mathematical worth through her own effort.

The most difficult problem in designing programming tasks was to devise projects which effectively illustrated the desired mathematical idea, without too many confusing layers of programming detail. As much use as possible was made of programs consisting of only a few instructions. For example, the drawing instructions FORWARD and BACKWARD were used to model integers. That is +100 was represented as FORWARD 100 and -100 became BACKWARD 100. By this means students were able to model arithmetic expressions, consisting of operations with integers, in terms of simple sequences of drawing instructions that moved the Turtle.

Simple programs also helped to flesh out what was otherwise meaningless notation for the students. Thus the puzzling expression, 'the mapping j on the real numbers defined by j: x —> 1/x' was turned into a LOGO program to do the mapping which consisted of three lines of code. In this particular case the student did not have to run the program. Merely reformulating the mapping in the more familiar (to her) LOGO instructions enabled her to see what a mapping was and what this one did. As in the integer project, once the program had been used to help understand the idea, it was abandoned in favour of conventional mathematical notation. The program provided a convenient jumping off point.

Another way that the programming was kept simple was by enhancing the programming language with new instructions. These were implemented in LOGO (by the author) and conformed to the same syntactic rules as the rest of the language. One example of such a system consisted of a series of programs that enabled the symmetry transformations of the rectangle to be studied by experiment. A transformation corresponded to an action carried out by the computer in response to the student's command, and a state corresponded to the resultant picture of the rectangle drawn on a display screen. This exemplified the distinction between a state and a transformation that had been troubling some students. Another suite of programs illustrated the multiplication and division of fractions. This implemented fractions as operators, used to transform the size of a picture displayed on a screen in the ratio given by the fraction. The program allowed the students to explore visually such ideas as inverse, multiplication and division of fractions, and provided some meaning for that puzzling rule about 'turning upside down and multiplying'.

Conclusion

The programming activity had a number of advantages and disadvantages. It emphasised the importance of explicit language, it illustrated certain key concepts such as angle and transformation and it provided splendid opportunities for problem solving and personal discovery. But it could also cause frustration because getting a solution to a problem could take a long time, it was not easy to design programming projects that addressed the students mathematical difficulties at the right level, and ease of access to the machine encouraged the students to use trial and error methods rather than systematic analysis of a problem. It proved possible to increase students' understanding of particular mathematical ideas and to improve their attitudes to particular topics. But the deep-seated anxiety about the subject as a whole, expressed by some of the students, proved impossible to change.

It was found that using the drawing facilities to provide visual illustrations of mathematical rules was worthwhile, but asking students to rework algorithms in programming notation was not a useful activity when the students knew the algorithms already, but did not understand their meaning.

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Place-Value and Decimals by Margaret Brown

Introduction: the items

The aim of this area of the CSMS work was to find whether children could meaningfully use the base-ten place-value notation for both whole numbers and decimals, in the sense of both understanding how it worked and applying it to appropriate situations.

The study concentrated on the area of decimals since most children in the 11 - 15 year age group obviously had a reasonably sound basic knowledge of whole numbers, at least as far as a few thousand.

There was throughout the work a major problem of trying to differentiate between the 'relational' understanding of properties of number, and a mere 'instrumental' ability to carry out a technique. This probably caused greater difficulty in this area than in any other of the CSMS studies. Although it was felt that this distinction was usually possible during an interview, it was often less feasible with the results of written tests.

It was decided that as little explicit reference as possible would be made to fractions, although clearly the decimal system rests on the basis of this idea. In particular no fraction notation was used (.3 was referred to throughout as either 'three tenths' or '3 tenths', never as $\frac{3}{10}$) and there were no items requiring direct conversion between the two systems, although obviously several questions required an implicit correspondence to be made.

Since the test was type-written, and will be printed, it was decided that the British Standard notation would be used i.e. 30 000 rather than 30,000, and 4.29 rather than 4·29. An attempt to explain the equivalences of these notations was made both in the interviews and written tests, but it seems likely that the former, although probably not the latter, did cause problems with up to 5% of weaker children.

An introductory practice item introduced the naming system for the places;

children who were unable to fill in the 'names' for tenths, hundredths etc. were supplied with the correct answer, which they could refer to at any time during the test. A pictorial representation (by area) of the relationship between hundredths, tenths and a whole unit was also supplied to try to explain the meaning of these 'names'. This was done interactively for the interviews and class-test trials, and in the case of the latter, all the explanatory information was left on the blackboard. (There was some evidence that children failed to use this information, at least after the first few questions.)

Methodology

A set of items, as described below, was drawn up to test various aspects of the understanding of place value and decimals. These were then used for interviewing 27 children of varying ability levels in years 1 - 4 in the secondary school. During the interviews the items were changed several times; those which did not fulfil their purpose were removed and some were added to tap ways of thinking which appeared in the interviews and which were not necessarily expected beforehand.

The final selection of items was arranged as a class-test which was tried out with a fourth year mixed ability group in a girls' school, a middle-band third year group in a boys' school and a top set of fourth year girls in a third school. Some minor changes were made as a result of these various trials, but in general the items appeared to work satisfactorily.

The first class-test was given in the Summer term to a sample of first to fourth years as follows:-

1st years (aged 12) : 170 from 5 different schools
2nd years (aged 13) : 294 " 9 " "
3rd years (aged 14) : 247 " 7 " "
4th years (aged 15) : 238 " 6 " "

The children in each year were selected so as to be representative of that whole age-group in respect of ability (as measured by non-verbal IQ).

The results for the class-tests were computer-processed, allowing the most common answers (both right and wrong) to each question to be coded. Thus data on common misconceptions can be reported as well as simple facilities.

The correlations between each question and every other one were calculated (using the ϕ-coefficient) and only about half the questions, those which correlated well with each other, were retained for the final analysis. (Many of the others do give useful information however, and are reported on below.)

This final group of questions were grouped into different levels of facility, the cut-off-points between the different groups being determined according to the following criteria:

(a) the correlations (ϕ) between questions within each group should be high,

(b) each group should represent as coherent a group as possible in terms of the mathematical ideas which seemed to be involved,

(c) the range of facitities covered by each group should be about 10-15%,

(d) the groups should be 'scalable' in the sense that the number of children who had succeeded in two-thirds of the questions in one group but not with two-thirds of those in all 'easier' groups should be minimised. (In the case of this test this figure was reduced to 7% of all children.)

The children were then assigned to 'levels' according to the highest group (or 'level') of questions they had succeeded, using the 2/3 criteria described above.

Results of the written tests: the levels

The levels are summarised below. A great deal of extra information on children's understanding was obtained in the interviews. Some of this is included in a forthcoming book: CSMS project (1980) Children's Understanding of Mathematics (11-16). John Murray.

A brief description of the levels is given in Fig. 1. The bands on the left show the spread of facility for items in each level for each of the years 1, 2, 3 and 4 in the secondary school. These facility bands frequently overlap, due to the fact that the relative difficulty of the items often changed from year to year. The groups of items were chosen so as to minimise overlap in years 3 and 4, since these results were judged to be more stable than those for year 1.

The group of questions which form each level are numbered as they appear on the test-paper.

Level 1: Place-value in whole numbers up to thousands

Typical questions in this group were:

		Facilities by year			
		1	2	3	4
10,b.i)	Ring the BIGGER number: 20 100 or 20 035	86	89	91	94
7)	This meter counts the people going into a football stand. 0 6 3 9 9 After one more person has gone in, it will read:	68	77	86	88
12.b)	Write down any number between: 4100 and 4200	69	80	85	88

The questions involved fairly straightforward examples of the 'naming' of places and the type of simple relationships between places which are exemplified in the examples quoted above.

It is perhaps also worth quoting some items which concerned whole numbers

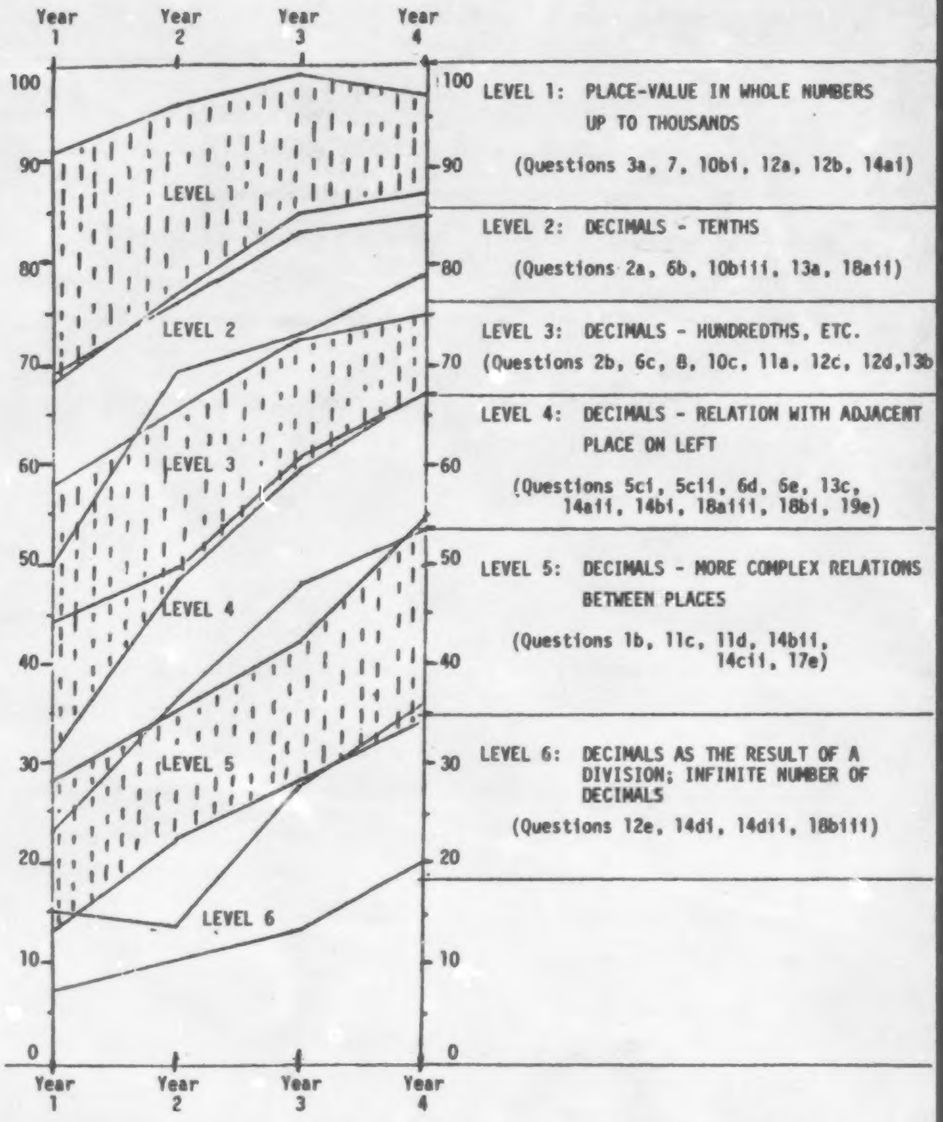


Fig.1 Range of facilities for questions in each level, over years 1 - 4

but were NOT in this level:

8) The number that is 2 less than 17 000 is ....

1	2	3	4
52	59	73	68

This question required either a number of relations between columns, or a good intuitive grasp of large numbers, or accurate use of the subtraction algorithm in the most complicated case. It actually fell into level 3 although it was not clear that it had more than a tenuous connection with the concepts involved in other questions in that level.

Another example which concerned whole numbers but which did NOT fall into any level as it had a poor correlation with all other items was:

16.b) Subtract

$$\begin{array}{r} 2312 \\ -\ \underline{547} \\ \underline{} \end{array}$$

1	2	3	4
61	61	62	66

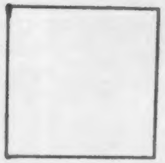
This was a straight 'formal computation' item, and unlike most items showed little improvement across the years, although there was a small improvement in year 4. It seems likely that children were 'triggered' into the formal algorithm, and the eventual result bore little relation therefore to the child's understanding of place value. (Further evidence of this is given from the interview data.)

A further item concerned with whole numbers was 1b. This fell into level 5 and will be discussed there.

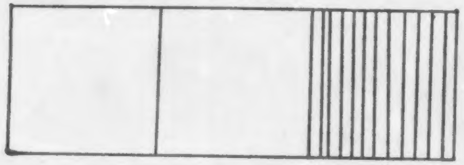
Level 2: Decimals - tenths

Typical questions in this level were:

13.a)



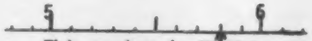
This is 1 square unit



1	2	3	4
69	76	81	84

The area shaded is [.] square units

6.b)



This number is [.]

1	2	3	4
62	74	83	85

10.b.iii) Ring the BIGGER number: 4.06 or 4.5

1	2	3	4
66	72	83	80

2.a) 0.2 The 2 stands for 2 ....
↑ ↑

1	2	3	4
63	70	73	79

These items were all concerned with either the straightforward naming of the tenths-place, or with the meaning of this as related to a visual representation. It can be seen that 6.b is found marginally more difficult than the others by first years, but rather easier by fourth years. This may well be due to increased experience of the reading of scales in science, as well as in mathematics.

Level 3: Decimals - hundredths etc.

Typical items include:

11.a) Six tenths as a decimal is 0.6

How would you write out a decimal:

three hundredths ....

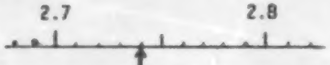
1	2	3	4
50	60	59	57

12.d) Write down any number between:

0.41 and 0.42 ....

1	2	3	4
37	49	66	71

6.c)



This number is $\boxed{\ .\ }$

1	2	3	4
31	48	66	71

Again it is clear that the 'scale-reading' item 6.c showed a much greater increase over the four years than did most other items. However the other 'interpolation' items which required the idea of the 'hundredths' as representing divisions between two adjacent 'tenths', had a similar pattern. The straightforward item 11.a which required naming, together with the use of zero as a place-holder, was found comparatively very much easier by first and second year pupils, who may have learnt this by rote, but surprisingly more difficult than the other two items by third and fourth year children.

Level 4: Decimals - relation with adjacent place on left

Typical examples include:

18.a.iii) Ring the number NEAREST IN SIZE to

$\boxed{0.18}$ → 0.1 / 10 / 0.2 / 20 / 01 / 2

1	2	3	4
44	48	61	59

14.a.ii) $\boxed{\text{Multiply by ten}}$

5.13 → ....

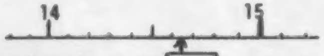
1	2	3	4
37	42	58	65

5.c.ii) $\boxed{\text{Add one tenth}}$

2.9 → ....

1	2	3	4
38	44	51	59

6.d)



This number is about $\boxed{\ .\ }$

1	2	3	4
24	36	53	61

Most items in this group simply required the idea that ten in one place would be equivalent to one in the next place on the left, or, perhaps rather more difficult, the notion in the first example above that where e.g. the hundredths digit was more than five, the number obtained by dropping off the hundredths figure was less close to the original number than that obtained by increasing the tenths digit by one.

Again the 'scale-reading' items were found relatively more difficult than the others by first and second years.

The second example, 14.a.ii, could have been done purely by rote with little understanding. In fact several children showed 'working', mostly for long multiplication. (Those who showed 'working' and arrived at a correct answer constituted 2%, 5%, 8% and 6% of the sample over years 1 to 4 respectively.) However the fact that the item correlates well with, and shows a similar pattern to, other items in this level suggests that on the whole children with little understanding did not manage to arrive at a correct answer even by rote.

The example 6.d is more difficult than 6.c in the previous level as it requires not simply counting divisions, but a conscious decision that there would be ten hundredths between 14.6 and 14.7 and hence the point half way along would be 14.65.

Level 5: Decimals - more complex relations between places

Typical examples include:

11.d) Four tenths is the same as .... hundredths

1	2	3	4
28	31	42	40

14.b.ii) Divide by one hundred

3.7 → ....

1	2	3	4
24	27	34	41

1.b) 5214 The 2 stands for 2 HUNDREDS

521 400 The 2 stands for 2 ....

1	2	3	4
22	32	31	43

This group of questions was rather diverse, but it included relations between two non-adjacent places (as in the second example above), as well as the ability to reverse the reasoning in level 4 questions to arrive at the fact that one in a given place is equivalent to 10 in the next place along on the right (as in the first example above). This latter process is that of 'borrowing' required in the subtraction algorithm, whereas the idea of 'carrying' as used in addition belongs in level 4. (In fact since the straightforward subtraction and addition computations did not correlate at all well with other items, it suggests that most children were doing them by rote and not necessarily understanding the processes underlying the algorithms.)

The third example seems at first rather out of place here, but essentially a not too dissimilar step is involved in going from the idea of 'twenty thousand' (i.e. 20 in the 'thousands' place) to '2 x ten thousand'. This makes it clear that the name of the place to the left of the thousands must be 'ten thousands', which is a fact surprisingly few children seemed to know.

The first example above embodies the concept of 'equivalent fractions', which also appears in the fractions paper and seems to be of a similar order of difficulty there.

Level 6: Decimals as the result of a division; infinite number of decimals

The items in this level were:

14.d) Divide by twenty

i) 24 → ....

ii) 16 → ....

Write 'NO' if you think there is no answer

	1	2	3	4
i)	9	13	28	34
ii)	7	11	25	36

18.b.iii) Ring the number you think is NEAREST IN SIZE to answer (do not work out the sum):

59 ÷ 190 → .003 / .03 / .3 / 3 / 30 / 300 / 3000

1	2	3	4
15	10	13	22

12.e) How many different numbers could you write down which lie between 0.41 and 0.42?

....

1	2	3	4
12	10	21	20

The second and third examples, 14.d.ii and 18.b.iii, showed that children were very unwilling to divide by a number larger than that with which they started, and in 18.b.iii tended to reverse the order of the numbers to make it 'sensible'. Even when the number was smaller, as in 14.d.i, there was uncertainty as to what to do with a remainder (those who correctly expressed the answer with a fraction in were also included in the 'correct' category).

The percentage 'correct' for item 12.e includes children who gave an answer like 'lots' or 'hundreds', even if it did not include strictly the idea that there were more than you could ever count. (7%, 7%, 16%, 16% of children in years 1 - 4 respectively gave an answer which conformed to this more exact criterion).

These results are extremely close to the proportions of children who were able to use the idea of 'infinitely many' in the context of graphs and fractions.

Fig. 2 shows the proportions of children in each year group who were judged to have attained each level.

The major point to notice is that all seven levels of understanding are found in each of the years 1, 2, 3 and 4 of the secondary school, although the proportions differ from year to year.

Thus in a truly mixed ability first year class of 25 children, about four children would be expected to have little understanding of place value even in whole numbers (i.e. level 0), whereas the number would have decreased to two in a similar fourth year class. In the same way about two children in the first year class, and six in the fourth year class, would be expected to understand most of the fundamental ideas of the decimal system.

By looking across Fig 2 it can be seen that provided children keep the same relative orderings, which is a non-trivial assumption, most children would expect to progress either one or two levels over the four year period.

One could generalise by concluding that the top 50% of pupils are likely by the time they leave school to have a reasonable if not complete understanding of decimals, whereas the lower 50% of pupils still have considerable gaps in their understanding. However this does not necessarily mean that they would not be able to cope in a more concrete situation where the decimals referred to measures of money, length etc.

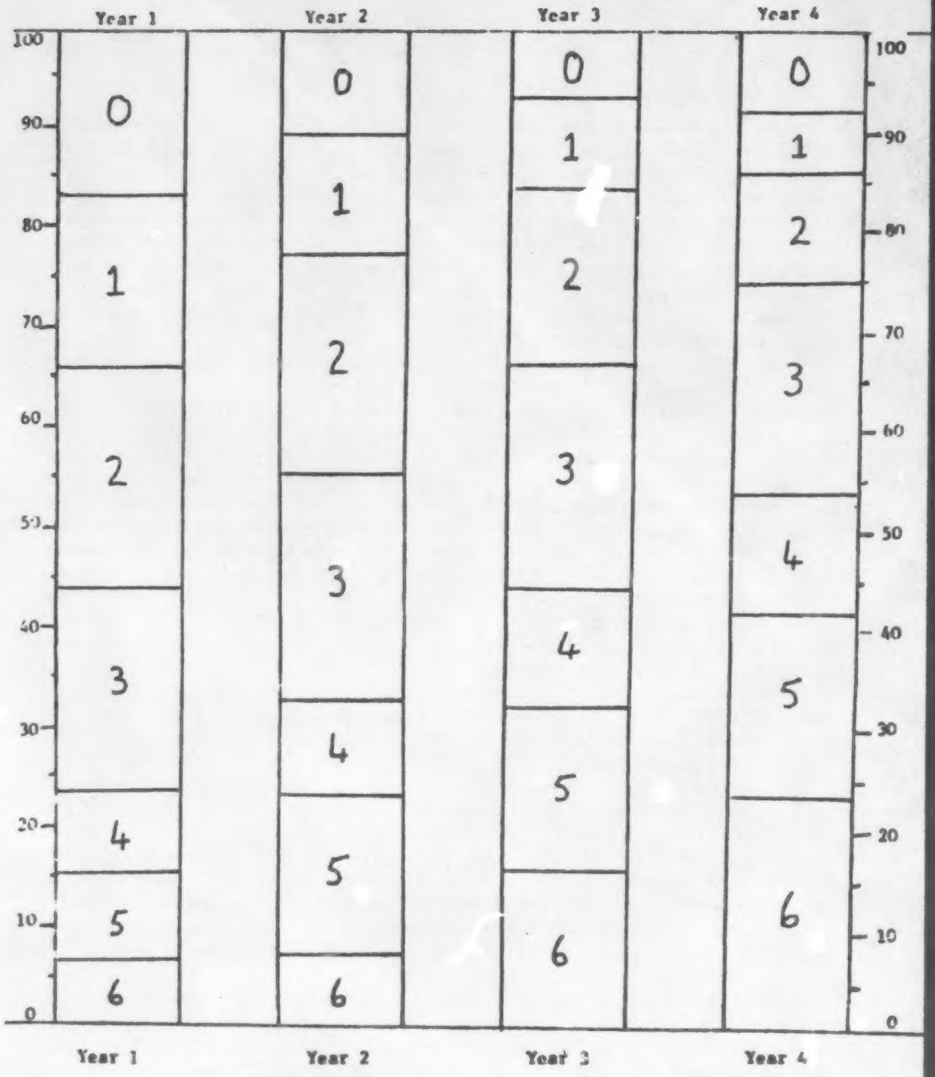


Fig.2 Percentage of children in each year of the secondary school who are judged to have attained each level. (The percentage labelled 0 did not attain any level).

The classification of problem-solving skills and procedures - Presentation of an Inventory

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Polytechnic of the South Bank,
London.

In 1978, a research project entitled "The Skills and Procedures of Mathematical Problem Solving in pupils of 9-13 years", supported by funds from the Social Science Research Council, was initiated at the Polytechnic of the South Bank, London. The project arose from the conviction of a group of mathematical educators[1] that mathematics is a process-based, rather than an information/skills based discipline, that the presentation of mathematics in schools did not reflect this evaluation, and that this mis-match could help to account for the poor quality of teaching and learning in the subject. In an attempt to improve upon the precision of definition of mathematics as a process-based discipline, the project first set out to identify and then to classify those skills and procedures which are displayed whenever mathematics is used in a problem solving mode.

It very quickly became clear that a distinction needed to be drawn between teachable/learnable components and 'abilities'. A range of abilities which ultimately must affect performance is apparent when any group of problem solvers is observed. However, it has also become apparent that, associated with ability is a set of techniques or approaches to problem solving, knowledge of which can also affect performance.[2] A useful analogy can be drawn with musical ability. Can aspects of problem solving be taught, just as the techniques of playing a musical instrument can be taught? In the process, it might be that the ability is improved to a high degree, or it might be that the learning becomes part of the general growth of the individual. It is not possible to make a concert violinist of every child but enough musical understanding and experience can be given to enable him to enjoy, appreciate and participate in music making. The project set out to determine how closely problem solving conformed to this model.

There has been growing interest among mathematical educators in problem solving, both as an activity in mathematics, as a mathematically transferable skill, and as a component of mathematical learning. At the International Congress on Mathematical Education held in Karlsruhe in 1976, nine communications pertained to problem solving in some way. On closer examination, however, a difficulty in the usage of the term becomes apparent. Some investigators are concerned to improve mathematical performance, and their mechanism for judging performance is what is termed the 'problem', more usually the sets of exercises, which are presented to pupils in the normal mathematics class, or text.[3] Although frequently termed 'problem solving', this kind of concern is with the pupil's efficiency in learning and reproducing the mathematics curriculum. The content of the problem is, consequently, prescribed and closed and the investigator is concerned to find ways of transmitting the appropriate information or skills so that the pupil's efficiency is increased. This is still in the context of content-based mathematics, where the learning of mathematics is viewed as the transmission of a body of knowledge, together with its appropriate range of techniques.

Many mathematical educators, however, reject this view of mathematics as being adequate to describe the discipline. Instead of regarding the subject as being content-based, it is preferable, in this view, to see the subject as being process-based, utilising such knowledge and skills as are required, together with a creative and enquiring approach, in order to make and to do mathematics.[4] Content-based mathematics draws, for its learning, on the past, already determined and stipulated subject matter. Process-based mathematics changes the emphasis from learning what has gone before, to making some mathematics happen now in order to develop some more mathematics in the future. The movement is from the passive recipient of pre-determined knowledge, to the active participant in determining what shall be learnt. The view of mathematics as a process is not new to mathematicians. Polya writes in his Preface to Mathematical Discovery, Volume), "If you have genuine bona fide experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is much more important than mere possession of knowledge ... What is know-how in mathematics? The ability to solve problems - not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity." He continues, "The teacher should develop his students' know-how, their ability to reason; he should recognise and encourage creative thinking - but the curriculum he went through paid insufficient attention to his mastery of the subject matter and no attention at all to his know-how, to his ability to reason, to his ability to solve problems, to his creative thinking."[5]

The problem of how to teach mathematics appears to be one of the unsolved problems of mathematics pedagogy. Is mathematics inherently so difficult that only a small percentage of the population can master the subject even to an elementary level? Is the subject inherently so unpleasant that it must arouse in those who come into contact with it, apart from a few masochists, feelings of aversion, distaste and hostility? Mathematicians cry 'no' to both of these questions but then how reliable are the affirmations of the converted? Why is it that the adjectives used by mathematicians to describe their subject include beautiful, interesting, exciting, pleasureable, while those who have rejected the subject speak in passionate terms of dislike, indeed, panic.[6] It appears that the problem is not only a mathematical, but also a pedagogical, one. A survey of the relevant research provided the research team with adequate justification to believe that there are now many well-founded insights into the nature of learning, and mathematical learning in particular.[7] From the point of view of problem-solving, the most convincing research is in the area of individual differences. What has become apparent is that every teacher is faced with an unwieldy gathering of individuals, each operating on a unique combination of multi-level differences. Just some of these differences are drawn out by, for example, the Piagetians[8] (developmental differences), Bruner[9] (differences in modes of thinking), Guilford[10] (differences in thinking pattern, i.e. convergence and divergence), Skemp[11] (different levels of understanding), Dienes[12] (differences in thinking strategies). Other differences become apparent although not yet substantiated by research. The pedagogic problem of coping with, never mind meeting the demands of, such complexity is overwhelming. Both teacher and students need a mechanism which allows learning to take place and which positively gains from the diversity rather than struggling to overcome or suppress it.

The research project, The Skills and Procedures of Mathematical Problem Solving, directed its attention to just that. It identified two main tasks. The first was the construction of an Inventory of mathematical problem solving skills and procedures, those behaviours which are demonstrated whenever problem solving is taking place and which, having been defined and classified, could be taught and learned. The second task was the designing and testing of a structured programme through which appropriate problem-solving skills and procedures, as classified in the Inventory, could be taught to, and learned by, pupils between the ages of 9 and 13 years.[13] It is the purpose of this paper to describe the methods by which the Inventory was constructed and tested, to provide a full description of the Inventory and to compare the Inventory with other heuristic lists in order to ascertain its generality and its flexibility as a tool both for classification and for teaching purposes.

The construction of the Inventory

The Inventory grew out of many hours of observation of children, students and colleagues working on mathematical problems. As already stated, these problems were not of the routine, or exercise, variety but were much more like mathematical puzzles or investigations. A range of such problems, including geometric, numerical, logical and materials-based problems were used and an attempt was made to create and use a schedule of favoured problem-solving responses. The items on this schedule were compared with the favoured responses of other solvers so that as broad a range of items as possible could be collated. The group of colleagues deriving this schedule then looked at what had been obtained and compared it with what possible problem solving behaviours could have been appropriate. In particular, variability of solution methods and problem extensions were sought so that a wider range of problem-solving skills and procedures was exposed. The Inventory grew out of this comparison and the and the procedure was then adopted again as the new listing was taken out and tested against the observable problem-solving behaviours of pupils. The present version ofthe Inventory was validated against the problem solving behaviours of mathematically achieving 16-year-old pupils in different schools across the country, in the expectation that they would display the most sophisticated problem-solving behaviours available to school pupils. On this validation, a combination of observation, interview and tape recording of the problem solver talking through while attempting his problem, was used. All three techniques presented difficulties. The Inventory which is now in use thus results from a combination of introspection and observation, constructed collaboratively by a group of colleagues.

The Inventory

The Inventory is presented in two sections, Skills and Procedures. Skills are described as teachable and storable for future use. They are reproduceable in the same form. They are conceived as static in the sense that they are acquired and stored. Examples of such skills would be the skill of writing, or the skill of drawing a graph. These abilities dominate the acquisition of skills: the ability to comprehend, the ability to transform information, and the ability to communicate. Procedures are described as methods of tackling a problem which move the problem into the domain where skills can be applied. In this sense, they are conceived as dynamic; procedures are enabling devices which push the problem from the realm of the unknown towards the realm of action by skills and consequent resolution. Procedures rely heavily upon affects such as confidence, curiousity, creative delight, spontaneity and interest. Abilities such as perseverence, and concentration and its inverse, decentration, are important. It is central to this approach that problems are not "solved" so much as "resolved" - problem-solving is seen as an ongoing process in which every 'new' problem has developed out of some previous problem although the forms of development are many. Each problem has embedded within it, therefore, the seeds of a new problem and part of the excitement of mathematical enquiry is the uncovering of these seeds and the consequent exposure of a new difficulty.

Problem-solving Procedures: these are listed under three headings, Entry, Attack and Extension.

Entry: the procedures by which the solver 'gets to grips' with the problem - the mechanisms for exposing the problem and making it amenable to attack.

1.1 Exploring the problem - non purposive juggling with data;
2 Trial and error - initial attempts at individual instances to extend understanding;
3 Defining terms and relationships in the problem;
4 Sorting and categorising information;
5 Ordering information.

Attack: the procedures for moving the problem towards a resolution.

2.1 Restating the problem (a) presenting the problem in a different way
(b) moulding the problem into a format amenable to a solution;
2 Working backwards from a known solution;
3 Trying related problems, e.g. Tower of Hanoi with 7 rings. Can it be done with 4?
4 Trying special cases;
5 Dissecting the problem,

6 Using numerical instances, i.e. when n = k;
7 Using empirical argument by (a) formulating hypotheses,
(b) testing a hypothesis by access to cases and considering each systematically,
(c) classifying the cases obtained,
(d) analysing the relations;
8 Systematic control of variables holding one constant while investigating the interaction of the others;
9 Elimination of paths, e.g. using contradictions or counter example;
10 Generalisation within the problem from special cases to a general case;

2.11 Upsetting "set".

Extension: where do we go from here? Increasing the solver's understanding of the problem, helping to place the problem in a known context or develop understanding of a new context.

3.1 Generalisation from previous problems to a clas s of problems;
.2 Finding isomorphic problems;
.3 Creating new problems.

Problem-solving Skills: these are listed under two headings, Representation and Information Analysis.

Representation: the skills of constructing and/or using different modes of presentation and choosing appropriately between them.

4.1 Linguistic (a) the skill of listening and grasping questions, problems and ideas expressed in speech;
(b) the skill of comprehending the written words used in a problem and extracting the significant aspects;
(c) the skill of representing ideas in words;
.2 Pictorial (a) the skill of interpreting a picture, etc., used in the statement of a problem;
(b) the skill of drawing or constructing appropriate pictures, diagrams, tables, graphs and other visual representations;
.3 Concrete (a) the skill of choosing among known conrete representations those appropriate to the problem;
(b) the skills of using and exploiting a known conrete representation;
(c) the skill of constructing an appropriate concrete representation;
.4 Symbolic (a) the skill of choosing a known symbolism;
(b) the skills of using and exploiting a known symbolism;
(c) the skill of constructing an appropriate symbolism;
.5 Translation The skill of translating from one of the above forms of representation into another;
.6 Modelling The skill of representing a concrete situation in mathematical terms and using the representation:
(a) abstracting mathematical features of the situation;
(b) choosing between known models;
(c) constructing new mathematical models;
(d) applying results back to situation.

Information Analysis: the skills of collecting, organising, analysing and presenting information.

5.1 Using representational skills (a) to identify the data of the problem;
(b) to identify the information then generated;
.2 Making explicit the knowns and the unknowns;
.3 Arranging information systematically;
.4 Presenting the data to highlight the most significant aspects.

S.P.M.P.S. Inventory of Problem - Solving Skills and Procedures

Procedures

Entry
1.1 Exploring
.2 Trial and error
.3 Defining terms and relationships
.4 Sorting and categorising information
.5 Ordering information

Attack
2.1 Restating
.2 Working backwards
.3 Trying related problems
.4 Trying special cases
.5 Dissecting
.6 Numerical instances
.7 Empirical argument
.8 Systematic control
.9 Elimination of paths
.10 Generalisation within the problem
.11 Upsetting set

Extension
3.1 Generalisation to a class of problems
.2 Finding isomorphic problems
.3 Creating new problems

Skills

Representation
4.1 Linguistic
.2 Pictorial
.3 Concrete
.4 Symbolic
.5 Translation
.6 Modelling

Information Analysis
5.1 Identifying data and information
.2 Making explicit
.3 Systematic arrangement
.4 Presentation

The Inventory aims, first, to be a descriptive tool. Within it should lie any strategy or technique which might be helpful to a problem solver. For each individual problem, a distinctive pattern of choice of procedures and skills will be appropriate. Some skills and procedures cluster. For example, when using some of the representational skills, it is hard to avoid 4.1 Linguistic, as well. The use of 2.7 Empirical argument frequently leads to the use of 2.9 Elimination of paths.

As well as being a descriptive tool of the problem-solving process, the Inventory aims to be a teaching tool. It has been the observation of many mathematical educators that pupils are "taught" mathematics but expected to "acquire" problem solving attitudes and habits. This expectation is rarely fulfilled and it is the belief of those involved in this project that aspects of problemsolving must be made overt and explicit and experienced at that level, in order to be learned. "We cannot rely on students' abilities to grasp the useful problem solving strategies, if they are not given explicit instructions on their use. With such instruction, heuristics do make a difference."[14] Furthermore, it is the belief of those involved in this project that appropriate problem-solving skills and procedures can become part of the normal problem-solving repertory of children from quite an early age and that the resultant positive experience they can have from learning mathematics in this way will effect their attitudes and motivation towards learning the subject.

A comparison of the Inventory with Polya's list of heuristics

Polya describes the aim of heuristic as "to study the methods and rules of discovery and invention".[15] He offers "a list of mental operations typically useful in solving problems; the questions and suggestions listed hint at such operations."[16] Polya's list is in four stages with an associated set of questions or suggestions.

First: Understanding the problem: What is the unknown? What are the data? What is the condition?
Suggestions: Draw a figure. Introduce suitable notation. Separate the various parts of the condition. Write them down.
Second: Devising a Plan : Do you know a related problem? Could you restate the problem?
Suggestions: Find the connection between the data and the unknown. Consider auxiliary problems. Obtain a plan of the solution. Related problems : more accessible? more general? More special? analogous? sub-problems?
Third: Carrying out the Plan: Have you checked each step? Is it correct? Can you prove it?
Fourth: Looking Back: Can you check the result? The argument? Check differently? See it at a glance? Use the result or the method for some other problem?

The first thing that is noticeable about Polya's list is how many questions and exhortations it contains. A colleague once remarked that Polya's model of problem-solving was beautifully rational and concise ... until you tried to make it work with a problem you could not do. Then the difficulties became enormous.' For example, finding the connection between the data and the unknown is frequently the heart of the problem. Once that connection is established, the problem might well no longer be a problem. In the hands of a teacher like Polya, his model becomes a flexible and applicable instrument. However, most problem solvers have lacked the gentle induction into the intricacies of each of the questions and suggestions without which the model is not workable.

Examine Understanding the Problem. Before the problem is understood, it must be accepted. Acceptance is partially cognitive and partially affective and is a combination of being able to enter the problem and having the skills to gather the necessary information and represent it in some way. Beneath Polya's first requirement, then, is Entry 1.1 Exploring, and, possibly, 1.2 Trial and error. As part of Understanding the problem, he requires 1.3 Defining terms and relationships, together possibly with 1.4 Sorting and categorising information and 1.5 Ordering information. The necessary skills of Information Analysis are certainly 5.1 Identifying data and information and, possibly, 5.2 Making explicit. Polya specifies Representation 4.1 and 4.2, Linguistic and Pictorial, and possibly 4.3 and 4.4, Concrete and Symbolic. The result of this examination leads to the following Inventory breakdown of Understanding the Problem:

Procedures - Entry 1.1, 1.2, 1.3, 1.4, 1.5
Skills - Representation 4.1, 4 .2, 4.3, 4.4
Information Analysis 5.1, 5.2

Now examine Devising a Plan. For Procedures, devising a plan is part of Attack and 2.1 Restating reflects one of Polya's questions: Could you restate the problem? 2.3 Trying related problems is another of Polya's questions: Do you know a related problem? The difference here is between offering the problem solver some helpful questions to ask of himself, and giving him specific experience in using the heuristic of trying related problems or considering auxiliary or related problems. Rather than instructing the problem solver to look for a related problem, the purpose of the Inventory is to draw attention to particular problems which require, for their solution, the use of the heuristic of analogy and to give experience in using that heuristic. So the problem solver is not just told that trying special cases, or using numerical instances might be helpful, he is given the experience in which precisely that heuristic is helpful so that he values its use in future.

The result of examining Devising a Plan shows the use of:

Procedures - Attack 2.1, 2.3, 2.4, 2.5, 2.6
Skills - Representation possibly 4.5 and 4.6

Carrying out the Plan is a combination of checking and proving. Neither appears as such in the Inventory. Checking is considered to be part of 2.7 Using empirical argument, but is particularly highlighted as one of what are being called the Housekeeping skills of Problem solving. Housekeeping skills are tactics which aid good problem solving, without being necessary or required. Such a tactic is, for example, only using one side of the paper when attempting any one problem and number the pages as you go. This is a tidy procedure. Another tidy procedure is to present the problem solution in such a form that it can be communicated to others. This requires both checking and presentation skills. A particularly important housekeeping skill relates to upsetting set (Inventory 2.11), vital to coping with problems of the 'Aha' variety.[17] This has been presented, in the housekeeping Guidelines, as "Read the problem, REALLY read it" and then "Read the problem again and try to think differently. Ask yourself "Am I thinking something that isn't there?" Then "When you have a good idea, write yourself a message. Then you won't forget what you did and how you did it".

Proving, also, does not appear as such in the inventory. Methods of proof do appear, e.g. 2.9 Elimination of paths, but the actual presentation of proof becomes the application of mathematics rather than of problem solving.

Looking Back is a reflection of Polya's interest in teaching and learning so he emphasises checking and filing the result. In the Inventory, this becomes Extension (3.1, 3.2, 3.3) and in the housekeeping Guidelines the question "Think, does the problem give you 'What if....?' ideas for new problems?" is posed.

Polya has other advice to offer problem solvers.[18] He specifies for example, Recognize: use definitions (Inventory 1.3, 5.1), Regroup: transform the problem (Inventory 2.1, possibly 2.11), Remember: known theorems and problems (Inventory 2.3, 2.4) and Supplement: introduce auxiliary elements (Inventory 5.1, 5.2).

It would appear from this comparison that the Inventory does, indeed, contain all of the heuristics suggested by Polya, while offering a more extensive range and more detail in observing the relationships between the heuristics.

Comparison with Schoenfeld's Schematic Outline of the Problem-solving Strategy

One other comparison might prove fruitful and that is with Schoenfeld's very useful outline. Schoenfeld's approach is very similar to that of SPMPS. "We must recognise that the ability to apply individual problem solving strategies when appropriate is a necessary but far from sufficient condition for effective problem solving. ... In the absence of a 'managerial strategy' -- an efficient means for allocating one's problem solving resources and for selecting the appropriate approaches to individual problems -- one cannot expect the mastery of a series of individual heuristics to result in substantially enhanced global problem solving pefformance."[19] Schoenfeld's process begins with Analysis requiring understanding the statement (Inventory 4.1), simplifying the problem (Inventory 1.3, 1.4), reformulating the problem (Inventory 2.1) He suggests the possible need to draw a diagram (Inventory 4.2) and to examine special cases (Inventory 2.4) From Analysis, he passes to Design where alternatives are examined (Inventory 2.3, 2.4, 2.6, 2.7, 2.8, 2.9). Exploration is suggested as the 'heuristic "heart" of the strategy' where essentially equivalent problems (Inventory 2.1, 2.2, 2.9), slightly modified problems (Inventory 2.5, 2.8, 2.10), and broadly modified problems (Inventory 2.3, 2.8) are considered. There is a flow relationship between Design, Exploration and Analysis allowing feed-back between all three states. Finally, Implementation and Verification complete the process.

It is noticeable that Schoenfeld also assumes Entry, starting from the point where the solver attempts Analysis of the problem. Yet, frequently, it is observed that solvers are unable to begin their analysis because they lack an entry point. The distinction made between heuristics and managerial strategy is a useful one which, again, draws on the process aspect of problem solving.

Both Polya and Schoenfeld emphasise that their interest is to enable students to develop flexibility of approach to the solution of well formulated mathematical problems. Equally both authors acknowledge the expected application and fascination of these approabhes when applied creatively both inside and outside mathematics.

Conclusion

Growing interest in understanding, describing and teaching problem solving as part of mathematics education can be justified both in terms of its mathematical validity and in terms of its pedagogical place. The view of mathematics as a process is well-established. Much progress has been made recently in identifying the heuristics of the process and their inter-relationships. Equally, much progress has been made in the psychology of learning mathematics and problem solving in the mathematics classroom offers an opportunity to teachers and students to develop the subject in an environment which is both mathematically and pedagogically sound. At least one positive outcome could be changed attitudes to the subject. Another could be the development of problem solving behuviours applicable outside as well as inside the discipline. Presented here has been an attempt to describe and classify the heuristics of problem solving by way of identifying teachable procedures and skills. In the context of a structured problem solving programme, the items of this Inventory can be stipulated and made available to students in an overt way so that they are encouraged to use, and re-use them. In this way, it is hoped that their problem solving behaviours will be improved so that, ultimately, they will display an increased interest, and ability, in solving problems.

Notes:

1. The group originally associated with the devising of the project and the early work work defining the approach now forms a Consultancy Group to the project. It consists of: Charles Cooke (Matlock College), Janet Duffin (University of Hull), Raymond Heritage (formerly St. Peter's College), Celia Hoyles (Polytechnic of North London), Derek Peasey (Hertfordshire College), Hilary Shuard (Homerton College) and Sr. Mary Timothy Pinner O.S.U. (Christ's College) , plus Leone Burton (now Director of the project at the Polytechnic of the South Bank).
2. These approaches have become known as the heuristics of problem solving since the term was revived by Polya, in How to Solve It, Princeton UP, 1945.
3. See, for example, Krutetski V. A., The Psychology of Mathematical Abilities in School Children, Chicago UP, 1976, and others
4. See, for example, Hadamard, The Psychology of Invention in the Mathematical Field, Dover, 1954, and Scandura, J.M., Problem Solving: a structural process approach with instructional implications, Academic Press, 1977.
5. Polya, G., Mathematical Discovery, Vol.1, Wiley, 1962, p.vii-viii.
6. Buxton L., An extreme reaction to mathematics, private circulation 1977
7. For the theoretical background to the pedagogical approach of the project see Burton, L.,; The Teaching of Mathematics to young children using a problem solving approach, to appear.
8. A full review of the developmental position is given in Donaldson, M., Children's Thinking, Fontana, 1978.
9. See Bruner, J.S., Goodnow, J.J. and Austin, G.A., A Study of Thinking, Wiley, 1956.
10. Guilford, J.P., Traits of creativity, in Anderson (ed), Creativity and its Cultivation, Harper and Row, 1959.
11. Skemp, R.R., Relational understanding and instrumental understanding, Mathematics Teaching 77, pp20-6, 1976.
12. Dienes, Z.P., An Experimental Study of Mathematics Learning, Hutchinson, 1963.
13. This programme is described in Burton, op.cit., to appear.
14. Schoenfeld, A.H., Problem solving strategies in college level mathematics, private circulation, p.14.
15. Polya, 1945, op.cit., p.112.
16. Ibid., p.130.
17. See Gardner, M., aha'. Insight, W.H. Freeman, 1978
18. Polya, G. Mathematical Discovery Vol. 11, Wiley, 1965, p.70.
19. Schoenfeld, op.cit. p.16

THE DEVELOPMENT OF ADDITION AND SUBTRACTION CONCEPTS IN YOUNG CHILDREN

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A major goal of mathematics instruction is to teach children to apply their mathematical skills to solve problems. It is frequently assumed that children must first master computational skills before they can begin to apply them to the solution of problems. However, it has been clearly demonstrated that children develop a variety of informal strategies for solving mathematical problems independent of instruction (c.f. Ginsburg, 1977; Resnick, Note 1). In fact many of the informal strategies are more sophisticated and demonstrate more insight than the formal procedures that are a part of instruction. This raises the hypothesis that, rather than depending on a prior knowledge of computational skills, simple problems may give meaning to basic mathematics operations.

The focus of the present program of the Mathematics Project at the Wisconsin Research and Development Center for Individualized Schooling is on children's concepts of addition and subtraction as reflected in their ability to solve problems representing addition and subtraction operations. Our working hypothesis is that outside the context of formal instruction many children develop strategies for solving a variety of different problems involving addition and subtraction operations. Furthermore, these strategies provide a basis for learning addition and subtraction concepts and skills. By identifying the processes that children use to solve problems at different stages of instruction and how differences in instruction affect the acquisition of these processes, our research is attempting to gain a clearer picture of how children learn basic concepts of addition and subtraction as well as to provide some insights into the development of their problem solving abilities. In the remaining sections of this paper, we shall discuss the general theoretical framework upon which these investigations rest and present results from some of the earlier stages of investigation.

Overall Plan of Research

The first major study in the project is a longitudinal study involving 150 children in eight classrooms. This study is attempting to trace the development of the strategies that children use to solve addition and subtraction problems from a stage preceding formal instruction through mastery of addition and subtraction algorithms. The study is also attempting to identify basic differences in the development of children and in instruction that leads to different patterns of learning of addition and subtraction.

Three major sets of variables will be investigated. The basic dependent measure is children's performance on a set of tasks related to addition and subtraction. One of the primary methods of determining this performance is through the use of clinical interviews. Other measures of children's performance on mathematical tasks will be obtained by short paper-and-pencil tests administered in a group setting.

The second set of variables deals with children's cognitive development. Two general sets of measures of development are currently being investigated. One includes certain Piagetian tasks that are hypothesized to be related to the development of children's arithmetic concepts and skills. These include measures of conservation, transitivity, and class inclusion. The second type of developmental variable being investigated is children's information processing capacity. We are currently focusing on the capacity of children's short term memory using measures based on the theories of Pascual-Leone (1970, 1976).

The third major set of variables deals with instruction and teaching behaviors. A key component of the observational scheme is the recording of pupil allocated and engaged time, using techniques developed in the Beginning Teacher Evaluation Study (Marliave, et al., 1977). Classroom observations of participating teachers and students will provide evidence as to what types of activities students have engaged in to learn addition and subtraction concepts, the amount of time they have engaged in them, and what teaching behaviors affect engagement.

In addition to the longitudinal study a number of shorter empirical studies are planned or in progress. These studies involve the development of instructional treatments and measurement instruments, the clarification of variables, the formulation of researchable hypotheses, and testing hypotheses from the longitudinal study.

Basic Addition and Subtraction Problem Types

Addition and subtraction problems can be grouped into two large categories. One category is the purely mathematical presentation of symbolic sentences in either horizontal or vertical form. The other is the set of verbal or concrete problem situations in which actions or relationships between quantities can be represented in terms of addition and subtraction operations. It is this second class of problems that is the focus of our research.

Three components of a problem have been identified. First, there are the two known numbers or the entities associated with, or representable by, these two numbers. The entities may be sets, objects, events, distances, durations, movements, or any measurable thing. Second, there is an operation, relationship, and/or action that connects together the three numbers or entities in some way. Third, there is the question or situation within the problem that seeks the identification of the unknown number or the measure of the third entity.

First problem component: Measurable entities. A distinction may be drawn between problems involving quantities that are discrete entities and problems involving measures of continuous quantity, like length, weight, volume, or age. In the first case, it is possible to represent the elements in the given problem set (apples, children, or whatever) by constructing an equivalent set by counting out the appropriate number of elements. For continuous measures, however, any set that might be constructed would represent the quantity in a very different sense. There would not be a one-to-one correspondence, because there are no identifiable elements of a problem set. Thus continuous quantities may present potentially more complex problem situations. It is also possible that continuous measures that are not directly observable like age or weight are more complex than measures like length or area that tend to be more easily discerned on a visual basis.

If the initial problem presentation is in the physical mode, then it is assumed that the child would have to initiate the measuring process to assign a number to the entities described in that problem. It is presumed that if a particular measurement skill is inaccesible then the solution process could not be carried to its end. If the verbal modes were used then it is likely that the actual measurements of the entities would be given as part of the problem presentation.

Second problem component: Contextual relationships or actions on entities. This second component stands out as having major significance. Together with the first component, it is amenable to representation in a variety of modalities. But what is perhaps even more important, it provides, along with the third component, the structure of the problem itself.

The second component has three dimensions. The first is the presence or absence of a set inclusion relationship. Next, it is possible to distinguish between a clearly implied action involving the entities as opposed to a static relationship. Finally, there is an order relationship that can be present. We shall amplify on these dimensions in the succeeding paragraphs. In what follows we shall let A, B, and C represent the sets or entities to which the numbers a, b, and c can be assigned. Further, a set or entity A' will be related to A by the fact that the measure of $A' = a$; similarly for any set or entity B' or C'.

One may consider the set inclusion or sub-set relationship between the entities. One relationship that is directly connected to addition/subtraction situations occurs when two of the entities are disjoint but are subsets of the third. Thinking of the mathematical models $a + b = c$ and $a - b = c$, these set-subset relationships can be represented by $A \cup B = C$ and $A = B \cup C$. A second, identifiable relationship that occurs within addition/subtraction situations is characterized by the fact that one of the entities may be a subset of, or included in a second entity but then the third entity is not included in the second or first. In set notation, this relationship would be $A \subseteq C$ but $B \not\subseteq C$ or $B \not\subseteq A$.

The second dimension is identified with some potential action or movement involving the entities. We can consider an explicit statement of action or movement causing a change in size or position of the entities. Greeno (Note 2) describes this as the Cause/Change schema; Steffe (1970) and LeBlanc (1971) talk of such situations as having a described transformation; Nesher and Katriel (Note 3) call this the Dynamic description. On the other hand, there can be a static situation involving the entities.

The third dimension involving the contextual relationships among the measurable problem entities is best described as an order relationship. In the static relationship among entities, there is the notion that one entity is larger or smaller than another. Where action is implied in the problem, that action results in something being made larger (increased) or being made smaller (decreased).

It is helpful to visualize this overall characterization of problem types by means of a three-dimensional solid that represents a two-by-two-by-two matrix. Such a figure is depicted in Figure 1.

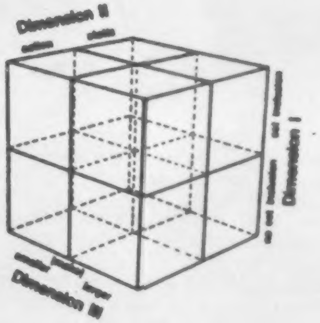

Figure 1. Schematic diagram of characteristics of addition/subtraction non-symbolic problems.

Each of the cells of the three-dimensional matrix can be exemplified by problem types that are generally well-known in the literature. We shall present a brief characterization of these, together with selected examples.

Joining. Joining is the process of actively putting together a set or entity B with a set or entity A to form a new single set or entity C. A is made larger or increased by B so that the union has measure c. Both A and B are subsets of C. It is often the case that once the union or join of A and B is carried out to form C, original sets or entities A and B are no longer distinguishable from each other.

Example 1. Wally has 3 pennies. His father gives him 6 more pennies. How many pennies does he have altogether?

Separating. Separating is the process of breaking up or separating a single entity into two sub-entities B and C and then taking away or removing one of those two. As with joining, it is usually the case that prior to separating a larger entity A into its two component sub-entities B and C in order to take away one of those two, those two sub-entities are not distinguishable one from the other. Greeno identifies the change direction as being a decrease. Others (e.g. Gibb, 1956) describe this problem as a "subtractive" situation.

Example 2. Sue has 14 pennies. She spent 11 pennies on a toy. How many pennies does she have left?

Equalizing. Equalizing is a problem type that is not as well-known as the others being discussed. It is used extensively in the earlier parts of the Developing Mathematical Processes (DMP) program that was developed at the Wisconsin Research and Development Center for Cognitive Learning (Romberg, et al, 1974). Consider two disjoint entities A and C so that $a = c$. Then some entity B is adjoined to A so that $A \cup B = C'$; that is, the union C' of A and B is such that it is equal in measure to the given entity C. This situation is known as an equalize-add on problem. A definite or implied action is carried out that is incremental in nature.

Example 3. There are 6 boys and 9 girls on the class basketball team. How many more boys will have to be put on the team so that there will be the same number of boys and girls?

A different equalizing situation occurs when $a \overset{>}{-} c$, in which case a subset B is separated from A so that the remaining set C' is equal in measure to C. This situation is known as an equalize-take away problem. Here the definite or implied action is decremental in nature in that entity or set A is made smaller in order to equalize it with C.

Example 4. There were 10 chairs at the table. Then 2 chairs were taken away so that there would be the same number of chairs as there were people to sit at the table. How many people were there to sit down?

Part-Part-Whole. This is a static relationship that exists between an entity having a particular attribute and its two disjoint but complementary parts. This relationship is isomorphic to the joining situation in that the set-notation representation is the same, namely, $A \cup B = C$. It is different from joining in that the two sub-entities will have some identifying characteristics that distinguish them from each other regardless of their position or any other relationship relative to each other. Greeno (Note 2) describes this type of problem situation as a Combination schema; Steffe (1970) and LeBlanc (1971) simply say there is no described transformation; Nesher and Katriel (Note 3) use the term Static description. We make note of the fact that the part-part-whole problem type--a static situation in which the set-inclusion relationship is present--does not appear to divide itself naturally along the third dimension of order. It is tempting to try to dichotomize the problem types on some logical basis. So far, such attempts have not been successful.

Example 5. Maria has some toy cars that are either red or blue. She has 4 red cars and 9 cars altogether. How many blue cars does she have?

Comparison. The static relationship of order existing between two disjoint entities is usually called the comparison (Greeno, Note 2; Nesher and Katriel, Note 4; Gibb, 1956) or difference situation. A comparison-larger problem situation involves the amount by which a larger of two compared entities exceeds the smaller on a stipulated attribute. In this situation, the student is directed by the problem language to focus on the fact that some entity A is larger than a second entity B. The two entities being compared are disjoint ones. The amount of difference, or excess of A over B is the number c, which is the measure of an entity C that is a subset of A. The relationship among A, B, and C is $A = C \cup B'$ where B' is an entity equal in measure to the given entity B.

Example 6. Mark has 2 balloons. His sister Connie has 7 balloons. How many more balloons does Connie have than Mark?

A second, similar type of problem is called the comparison-smaller problem. As with the comparison-larger problem, it involves a static relationship of order between two disjoint entities. The major point of departure is that the comparison-smaller problem focuses the attention on the smaller of the two entities and the amount by which it lacks being the same size as the second entity.

For each of these basic problem types there are instances where the problem can be stated in such a way that in the mathematical model--the sentences $a + b = c$ or $a - b = c$--the missing number would not always be in the "c" position. This is a characterization of the missing third number in a problem, the third component of a problem situation, to which we now direct our attention.

Third problem component: Characteristic of the missing number. A canonical situation is one for which the literal mathematical sentence representation would be a canonical sentence of the form $a + b = \square$ or $a - b = \square$. As an example, consider the joining problem, "Wally has 3 pennies. His father gives him 6 more pennies. How many pennies does he have altogether?" The model for this problem is the canonical addition sentence $3 + 6 = \square$ and the expectation is that the problem solver would likely impose an additive structure to the problem. On the other hand, if the conditions of the problem were changed slightly to, "Wally has 3 pennies. His father gave him some more pennies. Now he has 9 pennies altogether. How many pennies did his father give him?," the direct literal translation or representation would be the non-canonical sentence $3 + \square = 9$. This problem has now taken on a totally different structure from the first problem even though both involve the action of joining and both involve the number triplet 3, 6, 9. The second problem is an exemplar of the "missing addend" situation which has been traditionally thought of as a particular type of subtraction problem.

It is important to note that the suggestion of the canonical or non-canonical sentence model is one that we have adopted to make clear the distinction between types of problem situations. It should not be inferred that we expect young children to choose this particular type of symbolic modality as a way of representing a problem. In fact, for the missing-addend type a more mature child may correctly perceive the situation as one which is amenable to symbolic solution by direct subtraction. This would be particularly true if the numbers in the problem were larger than those given in the above example. Representing the total problem by the sentence $26 + \square = 53$ might not be as efficient as using $53 - 26 = \square$, especially if the latter were written in the alternative vertical form. Note, however, that using the latter sentence would no doubt involve some type of transformation on the student's part.

Problem Structure. Since student interpretation of problem structure plays a major role in determining how that student attempts to solve the problem, i.e., what sequence of representations and transformations he/she chooses, it is important that we attempt to clarify what is meant by problem structure. We suggest that it is the combination of the second and third problem components discussed earlier which gives a problem its structure.

A strictly mathematical interpretation can be given to a problem structure. Looking momentarily at symbolic problems, there are two possibilities, addition and subtraction.

Addition: $a + b = c$ Subtraction: $a - b = c$

When the third problem component--location of missing number--is considered, then the structure becomes more apparent. For the sake of the example, suppose the missing number is in the "a" position, giving

Addition: $\square + b = c$ Subtraction: $\square - b = c$

The mathematical structure is simply defined as what binary operation on the two remaining numbers is required to give the correct solution.

For some, the consequence of this analysis for instruction would be abundantly simple. That is, teach children to look at or analyze a problem situation in terms of deciding which binary operation to perform on the two given numbers, and once the correct analysis is made, to carry out that operation. However, as anyone who has worked with young children knows, things are not that simple. Children are not, and do not necessarily think like mathematicians or even like mature adults who are not mathematicians. Children interpret problems within the limitations of existing mental capabilities or schemata. Thus it seems that we must also consider a psychological structure of problems and the degree to which it maps directly on the problem's mathematical structure.

We suggest that the psychological structure of addition and subtraction problems is a relative matter. For many of the variations--canonical vs. non-canonical--of the different problem types that were represented in the three dimensional two-by-two-by-two matrix, the psychological structure appears to be unambiguous in that almost all children treat those problems in the same way. For example, the canonical joining problem alluded to earlier about Wally and his pennies maps almost perfectly onto the mathematical canonical addition problem. This is true whether it is Wally or Fred or Mary or whether the objects are pennies, candies, or pet frogs or whether the number triplet is (3,6,9), (5,7,12), or (2,8,10).

For other problem situations the psychological structure imposed by children is not so obvious. Furthermore, one can reasonably expect the child's imposition of structure to change as he/she changes developmentally and as the result of direct instruction. It is also possible that various semantic differences in the statement of non-symbolic problems could produce some differing interpretations by children. So, too, could relative number size. For example, in a non-canonical part-part-whole situation involving the number triplet (1,8,9) the use of the given numbers 1 and 9 might initiate a different problem solving response than use of the numbers 8 and 9.

Some Preliminary Results

In this section, we present results from one pilot study conducted in February 1978 and from the first two interviews of the longitudinal study that began in September 1978.

Pupil Behaviors and Strategies

Addition strategies. Three basic counting strategies for solving addition problems were found in this study. Several strategies that were not based on counting were also identified.

(CA) Counting all - The child counted the complete union of the sets represented in the problem.

(CF) Counting on from first number - When counting cubes, fingers, or mentally, the counting sequence began either with the first (smaller) given number in the problem or the successor of that number.

(CL) Counting on from larger number - The counting sequence began with the larger (second) given number or with the successor of that number.

(KF) Known fact - The child gave an answer with the justification that it was the result of knowing some basic addition fact.

(H) Heuristic - Heuristic strategies were employed by a few children to generate solutions from a small set of known basic facts. These strategies usually were based on doubles or numbers whose sum was 10.

Subtraction Strategies. Four basic subtraction responses were identified. They take on a different form depending upon the mode of representation chosen. For concrete representations they are:

(S) Separating - The child models the larger given set and then takes away or separates, one at a time, a number of cubes equal to the given number in the problem.

(ST) Separating to - After the larger set is modeled, the child removes cubes one at a time until the remainder is equal to the second given number of the problem.

(AO) Adding on - The child sets out a number of cubes equal to the smaller given number (an addend). The child then adds cubes to that set one at a time until the new collection is equal to the larger given number.

(M) Matching - The child puts out two sets of cubes, each set standing for one of the given numbers. The sets are then matched one-to-one.

Three more abstract counting strategies were also observed. These are the analogues to the first three concrete strategies listed above.

(CB) Counting back - A child initiates a backwards counting sequence beginning with the given larger number. The backwards counting sequence contains as many counting number words as the given smaller number. The last number uttered in the counting sequence is the answer. This is the counting analogue to the separating (S) strategy.

(CT) Counting back to - A child initiates a backwards counting sequence beginning with the larger given number. The sequence ends with the given smaller number. By keeping track of the number of counting words uttered in this sequence, the child determines the answer to be the number of counting words used in the sequence. This is the counting analogue to the Separating to (ST) strategy.

(CU) Counting up from smaller - A child initiates a forwards counting sequence beginning with the smaller given number. The sequence ends with the larger given number. Again, by keeping track of the number of counting words uttered in the sequence, the child determines the answer. This is the counting analogue to the Adding on (AO) strategy.

The known fact (KF) and heuristic (H) categories follow the same rules as the corresponding addition categories.

The First Pilot Study

Ten problem types were employed. The three problems with an additive structure were a Joining, a Part-Part-Whole, and a Comparison with the smaller set and the size of the difference given. The seven subtraction problems included a Separating, a Comparison, a missing addend Part-Part-Whole, two missing addend Joining, an Equalize add-on and an Equalize take-away. For each of the 10 problem types a verbal problem and a problem involving action or relationships between sets of cubes were generated. Forty-three first grade subjects (age 6 or 7) were given each of 20 problems individually administered in two 20-minute interviews. The interviews were carried out in February 1978 prior to any formal instruction in symbolic representation of addition and subtraction problems. All problems were read to the subjects and repeated as necessary.

Results on the set of concrete problems are less interesting and will not be discussed in any detail. Strategies children used seemed to be determined mainly by the characteristics of the sets of cubes they had presented to them when they began to solve the problem.

For the verbal addition problems the Joining and Part-Part-Whole problems tended to be easy with a high degree of success, at least in terms of the choice of an appropriate strategy. The patterns of solution tended to be very similar. What errors did occur were mainly as a result of miscounting. The Comparison problem proved to be extremely difficult; subjects appeared to misunderstand the context of the problem and many gave as their answer the stated difference that was part of the original problem. Most subjects used cubes as models for solving. Almost all subjects who did use cubes used a counting all (CA) strategy. When fingers (F) or no physical model were used then the children resorted to the more efficient counting on from first (CF) or counting on from larger (CL) strategies. A relatively small number used a known number fact (KF) or a heuristic (H) strategy.

Responses and strategies used for the verbal subtraction problems proved quite interesting; they are summarized in Table 1. Not shown in Table 1 is the number of correct responses. For the most part children were correct, at least with their choice of an appropriate strategy. As with addition, there was a sizeable incidence of counting errors. The important result to be noticed from the data is that problem structure does appear to be the major determinant of the solution strategy.

The Longitudinal Study

From the ten problems given in the pilot study, a subset of six verbal problems was selected for inclusion in the first set of individual interviews for this study. These are the canonical Joining, the canonical Part-Part-Whole, the canonical Separating, the canonical Comparison, the missing addend Joining, and the missing addend Part-Part-Whole. These problems appear at four difficulty levels in one interview (broken into two administration periods separated by at least one day) according to a two-by-two factorial design. The factors are number domain--size of the sum of a number triple being either between 5 and 9 (b) or between 11 and 16 (c)--and the presence (+) or absence (-) of manipulative devices to assist in solving.

TABLE 1

SUMMARY OF VERBAL SUBTRACTION PROBLEM STRATEGIES

Problem	Strategies*								
	S	ST	AO	M	CB	CT	CU	H	KF
Separating	19	0	0	4	9	0	0	3	2
Joining 1	4	0	15	2	1	0	2	3	0
Joining 2	4	1	14	5	0	0	8	3	0
Part-Part-Whole	9	0	10	4	3	0	3	0	2
Comparison	8	1	2	17	0	0	3	3	1
Equalizing +	9	2	4	15	0	1	1	5	0
Equalizing -	14	4	0	13	1	3	2	2	0

*See text for explanation of symbols

Each of the four levels of six problems has different, but equivalent, wording and differing number triples randomly distributed across subjects. A sample of 150 first-grade children from eight classrooms is being used. Interviews begin with the easiest set--small number domain with manipulatives present (b+)--and are terminated early if it is apparent the subject will be unable to generate any codeable response for subsequent problems.

Data is currently available from the first interview given in October 1978 and the second interview given in January 1979. Both interviews came prior to any formal instruction in symbolic representation of addition and subtraction. Results on the two addition problem tasks are presented in Table 2, and on the four subtraction problem tasks in Table 3.

The single clear impression gained from inspection of the two addition tasks is the strong similarity of the pattern of responses which suggests that children do not impose different structures on these two tasks. As expected the counting all (CA) strategy predominates when the numbers are small (b+ and b-) and when manipulatives are available for modeling the numbers (b+ and c+). The other notable result is the marked shift in the second interview towards the more efficient and sophisticated strategies of counting up, and use of number facts and heuristics with the smaller numbers.

The subtraction results are similar with those of the pilot study in that problem structure seems to be the strongest determinants of the choice of strategy.

These data give an overall picture of problem structure and problem difficulty and the effect of number size and availability of manipulatives on solution strategies and success. They do not characterize individual students or clusters of students. Further analyses are required in this regard.

TABLE 2

Addition Task Results
Number of First-Grade Children Interviewed in October 1978 (January 1979)

Task	Strategy*	Difficulty Level* b+	b-	c+	c-
1. JOINING	CA	70 (83)	60 (52)	67 (80)	29 (26)
	CF	13 (7)	11 (15)	5 (12)	11 (25)
	CL	9 (13)	8 (14)	11 (18)	12 (22)
	KF	14 (25)	10 (26)	1 (2)	0 (1)
	H	0 (4)	8 (4)	3 (6)	2 (7)
n**		144 (150)	118 (138)	100 (127)	90 (111)
# correct		96 (126)	90 (113)	68 (99)	44 (68)
4. PART-PART-WHOLE	CA	84 (79)	62 (52)	74 (73)	25 (28)
	CF	7 (7)	8 (13)	4 (15)	12 (12)
	CL	8 (12)	13 (14)	5 (19)	11 (25)
	KF	8 (20)	11 (27)	2 (3)	1 (4)
	H	4 (9)	5 (8)	2 (5)	1 (6)
n**		144 (150)	118 (135)	100 (124)	76 (102)
# correct		108 (123)	92 (108)	72 (107)	41 (68)

*See text for explanation of symbols

**n = actual number of students administered the task

TABLE 3

Subtraction Task Results
Number of First-Grade Children Interviewed in October 1978 (January 1979)

Task	Strategy*	Difficulty Level* b+	b-	c+	c-
2. SEPARATING	S	72 (93)	53 (60)	81 (102)	33 (26)
	CB	6 (5)	7 (5)	2 (1)	6 (9)
	KF	7 (17)	9 (15)	1 (2)	1 (2)
	H	1 (5)	5 (4)	1 (3)	1 (2)
n**		144 (150)	118 (138)	100 (127)	90 (111)
# correct		92 (108)	74 (92)	60 (92)	24 (31)
5. COMPARISON	S	21 (18)	10 (9)	18 (12)	2 (1)
	[illegible]	18 (36)	2 (4)	26 (45)	0 (0)
	AO	10 (4)	16 (16)	6 (5)	5 (9)
	CU	9 (13)	17 (22)	9 (13)	18 (26)
	KF	4 (7)	6 (13)	1 (2)	1 (3)
	H	1 (3)	2 (1)	1 (1)	3 (6)
n**		133 (147)	111 (133)	97 (123)	61 (89)
# correct		75 (84)	61 (76)	55 (61)	29 (42)
6. JOINING MISSING ADDEND	S	6 (2)	0 (0)	7 (3)	0 (0)
	AO	60 (68)	53 (43)	45 (65)	10 (16)
	CU	15 (24)	28 (24)	13 (18)	26 (39)
	KF	7 (23)	9 (25)	2 (3)	1 (2)
	H	2 (4)	2 (3)	2 (6)	2 (3)
n**		128 (147)	111 (133)	97 (123)	55 (88)
# correct		92 (120)	86 (110)	56 (86)	36 (59)
3. PART-PART-WHOLE MISSING ADDEND	S	41 (45)	32 (28)	55 (67)	19 (12)
	AO	10 (3)	6 (6)	7 (6)	4 (2)
	CU	7 (7)	5 (7)	3 (5)	11 (16)
	KF	5 (12)	3 (19)	0 (3)	0 (1)
	H	2 (5)	4 (4)	3 (3)	2 (4)
n**		144 (150)	118 (138)	100 (127)	90 (110)
# correct		67 (82)	50 (71)	51 (68)	19 (34)

*See text for explanation of symbols

**n = actual number of students administered the task

Conclusion

The purpose of this paper is to provide the reader with a brief overview of the theoretical framework supporting the current research program of the Mathematics Project of the Wisconsin Research and Development Center. The results presented must be considered as tentative. Yet they suggest certain possible conclusions.

A striking result of our research to date is the high level of success of first-grade children in solving verbal problems prior to formal instruction. Children are not only successful in modeling action or relationships implied in problems. They are also able to use different models of addition and subtraction when convenient.

The first-grade children in our studies give very little evidence of the types of systematic errors reported in previous studies. Very few of them used the wrong operation in their solutions. Since this error has been observed primarily with older children who have already experienced formal instruction in addition and subtraction, it may actually be a result of learning symbolic representations. Typically addition and subtraction are introduced in terms of joining or separating sets using either pictures or concrete objects. Then children are drilled on abstract problems with number sentences. When they finally get to verbal problems, their response is, "Is this a plus or a takeaway?" Because the operations are initially learned outside of the context of verbal problems and children are simply told that addition and subtraction can be used to solve these problems, they have no basis for using their natural intuition to relate the problem structure to the operations they have learned. In other words, their natural analytic problem solving skills are bypassed, and they too often resort to relying on superficial problem characteristics to identify the correct operation. This may result not only in a superficial concept of addition and subtraction but also in a decline in general problem solving ability.

The tremendous variability between and within children in the solution processes used suggest that before receiving formal instruction, young children do not transform problems into a single type and apply a single strategy. The results indicate that children have available a rich repertoire of strategies and that they make use of many of these to solve various problem types. It is still not clear what triggers the use of a particular strategy; but it seems plausible that children solve each problem type directly, rather than collapsing them and applying a single strategy consistently.

The picture painted by this desciption is quite different from that proposed by Greeno (Note 2). In Greeno's description, the limiting factor is the number of different solution strategies which children have available. Since empirical data show that children can solve a variety of problem types it was assumed that they must transform them in order to successfully apply the few strategies which they have acquired. The results presented here suggest that even prior to instruction most children possess the different strategies which are required to solve each problem type directly. No transformations are needed. In fact, it may be the transformation process itself which is the limiting factor as children begin instruction.

Arithmetic instruction frequently illustrates a particular operation such as subtraction with several problem types (eg., Separating, Part-Part-Whole, Comparison). Although most children can solve each type of problem using an appropriate strategy (e.g., separating, add on, matching), they may have trouble transforming these problems and understanding that a single strategy would be appropriate for all of them.

The results of our research support the hypothesis that verbal problems may be the most appropriate context in which to introduce addition and subtraction operations. Clearly, verbal problems are a viable alternative to traditional settings since children are able to interpret them and generate solutions prior to formal instruction. Verbal problems also provide different interpretations of addition and subtraction, interpretations that are important for children to understand. Perhaps by introducing operations based on verbal problems and integrating verbal problems throughout the mathematics curriculum, rather than using them as an application of already learned algorithms, children will develop their natural ability to analyze problem structure and will develop a broader conception of basic operations.

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Machines as medium for verification

1. Preface

In the following contribution I try to give an explanation, of why many pupils so easily make algebraic mistakes in solving mathematical tasks and why this weakness couldn't be put to an end with decisive success before now.

In order to solve a problem in mathematics (e.g. task of construction in geometry, writing of programs for computers, term-transformation, proof of theorems) the pupil applies any concepts, formulas, procedures or theorems to mathematical objects. From a sufficiently abstract point of view one can describe this situation as follows: a mathematical instrument is employed to any mathematical object. The mathematician would say, that a function is applied to an object. In the following sentences the concept of "application" will be analysed. Each of these instruments (operations, procedures, formulas, theorems) is defined for a more or less limited domain of application: theorems are only valid for certain assumptions, formulas have a field of validity, methods (functions) have a domain.

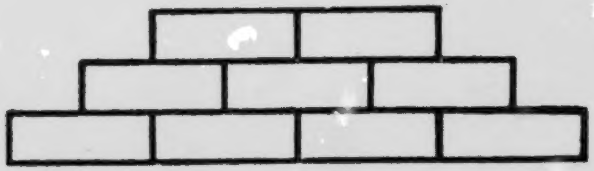
We consider first the correct application of such an instrument. Instruments are mostly applied with a distinct aim: a wanted term shall be obtained, an assumption for further applications of a theorem or a rule is to be created, results of constructions are desired. It is a widespread weakness (not only of pupils) to make statements which they are not entitled to make as a result of a use of instruments, either they have used that instrument in a wrong situation or if they had used that instrument correctly, they would have obtained a different result (wishful thinking). The pupil is obviously not educated to render an account of whether his chosen instrument may be applied in the given situation nor to disregard the aim of application of instruments in the process of production of the result, which he hoped to obtain. Overall one can say that we are concerned with the problem of correctness of operator-application, and this problem has two sections: the reason for error could be, that operations are specified incorrectly, or that the scope is not taken into account. A non-mathematical examp-

le will illustrate this situation once more: I remember that my Latin-teacher wrote under my translation "you must not conceive a grammar in order that the sense results in the translation you are supposing, but you have to apply the usual grammar in order to state which sense is meant".
In mathematics, application of a function to an argument does not cause any difficulty. Also in the foundations of mathematics the application of a computable function to its arguments is not much analysed. Only in combinatoric logic (cf. Curry 1963) has this question been analysed in more detail. Here the phenomenon "application of a function" has been made a basic principle of theory.

2. The Problem of Application of Functions

From the psychological point of view both problems mentioned above have been little examined. In connexion with research on programmed lessons there are some investigations which can be interpreted as preparatory works for such an examination. Here we refer to it very briefly. In extensive investigations of the problem of "setting" in mathematic lessons by Viet (Viet 1978) data of intelligence factors, personality factors (hope of success, behaviour of avoidance of effort), work at school, have been accumulated in mathematics in different fields for more than 2.000 pupils at the age of 10 in a study lasting three years. The aim of this project was to examine how far the differentiation problem in mathematics lessons can be solved by inner differentiation within the class using programmed lessons. In connection with this, a test of school achievment has been developed by Rollett and Bartram (KLPI) which aims to measure the qualification of pupils for programmed lessons by a subtest VBA (understanding and following of verbal instructions)(Rollet/Bartram, 1973). Here are two typical tasks of the subtest:

①



Try to find out the most central box and write down inside the initial letters of the following words. Pay attention please: first you must put them in order.

autumn
winter
summer
spring

② 9 13 25 2 12 74

If the fourth number is greater than the third number, then follow the second instruction. If the third number is greater than the fourth number, then follow the third instruction. But if the last number is greater than the third and fourth together, then follow the first instruction.

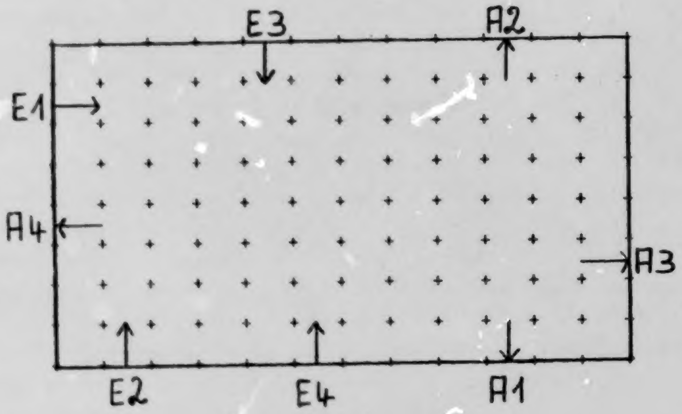
1st instruction:	draw circles round the number before the last and round the second number.
2nd instruction:	underline the first and the last number.
3rd instruction:	cross out the second and the fifth number.
4th instruction:	connect the number next to the last and the last number.

These investigations by Viet have proved that this subtest makes one of the best predictions for success at school in mathematics even in non-programmed form. This has been born out in my own examinations of lessons in which subjects were solving-problems in the development of networks of automata and algorithms (Cohors-Fresenborg 1978).
There one examined, how far the abilities described by the subtest can be devided into two sections:
understanding the instructions and correct application or practice of the instructions. This has not been done before. It is striking that the more difficult tasks of this subtest (which means these which are hardly ever solved) are also just longer or more complicated in its linguistic formulation, and that the instruction which must be applied is not necessarily more complex. "Complexity of the instruction" has not been examined or defined until now. The task now is to construct a test in which the complexity of the instructions depends very little on the complexity of the text which describes the instructions.

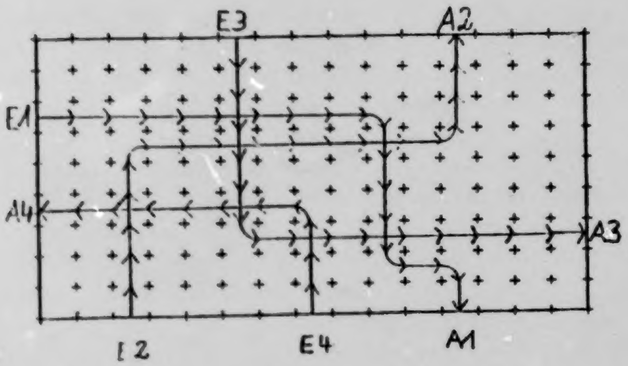
In this sense tasks of the following kind are suitable (Cohors-Fresenborg et al. 1979, Booklet 2):

Task:

Connect entrance E1 with exit A1
Connect entrance E2 with exit A2
Connect entrance E3 with exit A3
Connect entrance E4 with exit A4



Solution:



It would be interesting to examine the cognitive structure which is necessary to work such tests. The question of representation of these instructions is especially interesting: does the ability measured by this test perhaps consist in the ability to sequentialize these instructions and to work them out in sequential form? Such a sequential accumulation and using up can take place in two different ways: on the one hand by accumulating and transforming linguistic instructions in a suitable separated series and on the other hand by concentrating and sequentially accumulating the instructions of signs.

In didactics of mathematics it is known in another context that it is important for the pupil to put tasks in order (e.g. problems expressed in verbal forms) and to learn methods for that.
The simplex-method for arithmetic word problems propagated by Breidenbach (1971) and Bauersfeld (1965, 1969) consists in giving pupils a syntactical method (independent of the given task) to sequentialize the relations between magnitudes, hidden in the text, in a serie of calculations-operations. A similar method is offered by Winter (1972).

The most extensive research on algorithms and control processes inconnection with problem-solving was done by Landa (1969). His investigations show the importance of algorithmic methods for the improvement of understanding in a mathematical or linguistic context.

These examples can be interpreted as follows that, indeed, an abstraction of concrete task-formulation is done, but the aim is a network of operations, which can be handled syntactically. The result of this abstraction is the creation of new objects with which one is occupied in a concrete manner.
But there is a further problem for many pupils which must be examined in more detail. Take for example the case of a pupil who doesn't know how to multiply two fractions in a given situation. By the questions of the teacher and by an adequate hint he is able to name a rule "numerator multiplied with numerator and denominator multiplied with denominator", but he can't apply this rule. Questioning shows that he knows, indeed, what is a numerator and a denominator, but still he doesn't know what to do. He has got the right instrument but he can't apply it. Here one can object that he didn't understand fractions and therefore he can't apply the rule. This is not necessarily so, a mathematician can apply formulas, even if he doesn't know their proofs. Obviously he has got a kind of ability to operate correctly in the syntactical field without refering to the meaning.
This will be seen more clearly in the following example: in an investigation, a package of programs for a computer is put into the computer for a statistical analysis. In the description the meaning of variables has been explained but not the algorithm executed by the computer. The writing of corresponding control-cards is trivial, the execution of the program by the computer is certain, if the data is acceptable for this program.
It is not surprising, the problem of the application of a given algorithm in a given situation, has not been analysed by the methods used for computer-simulations of cognitive processes: the computer is marked out by the fact that it executes given programs for given acceptable data without error.

3. The problem of testing for correctness

Here we come to the second problem which occurs in connexion with the application of instruments: the investigation whether an instrument which is at issue in a given situation may be applied at all. Obviously many pupils don't ask a question about the admissibility of application of a formula. How can one explain that a considerable fraction of pupils apply completely unscrupulous algebraic formula and transformations when this isn't at that moment the subject of the present lesson, although he isn't unscrupulous in a normal linguistic context? In other words: what is the reason that many pupils answer to the teacher's objection that $A^2 + B^3$ is not AB^5, that they didn't understand the transformation. On the other hand they never say to their classmates that they didn't understand a foul in a game, but they apologize because they had forgotten a rule, or had applied a rule incorrectly. The readiness and ability to apply rules or instruments correctly, obviously depends strongly on the context and on the language, in which they are formulated. Many don't seem to be ready to think about this in the field of school-algebra. An explanation could be that the observed results of application of instruments (or of testing) touches the pupil's emotions less strongly. It seems that a special perception-algorithm for correctness-testing regulated by the emotions must be brought into play. In a certain context or with strong impulses this seems to take place imperceptible. Obviously the context school-algebra doesn't belong to these for most of the pupils!

In the usual sense it is not a problem of the contradiction "abstract" and "concrete". We all know people we would certify that they are able to argue in a very complex and logical way and we would still believe that they are not able to work on a formal problem (for example a task with variables). Even if we have to interpret a considerable part of inability as unwillingness we must still say that people obviously react differently to a logical problem, according to the language in which it is formulated. The decisive difference seems to consist in whether the argument is linked with meanings or whether it refers nearly exclusively to symbols or algorithms.

In this connexion another problem has to be considered: what is the meaning of "abstract"? The persons just described would call a transformation of algebraic expressions an abstract action. I think that we have here a misleading and dangerous application of the word "abstract". These people would argue abstractly about principles of philosophy, moral, aesthetics. In such a context the term "abstract" is suitable. On the other hand, transformations of algebraic expressions are essentially more concrete: they use to a large extent the shape of formal notation. Such formal expressions can be written down, they are visible. In this connexion they are much more to be regarded as something concrete: an X written on the board with chalk can make a white mark on your dark dress. For concrete handling with algebraic transformations no abstraction at all is necessary. In the problem treated here the contrast between abstract and formal is a much more suitable basis for an analysis. It might be that it has something to do with the problem, that certain pupils regard handling symbolic mathematics as something abstract, in the sense of something abstracted, and therefore they always have a need to fill this situation with meanings. As they can't find this meaning, they are not ready or able to go on calculating with concretely given series of symbols, irrespective of their meanings. Readiness and ability to employ the algorithm of correctness-testing mentioned above, seems to be dependent upon language, context and socialization. In children's play this algorithm is employed most of the time. Here there is an understandable context of actions, even if the game (e.g. draughts or game of cards) is, in a mathematical sense, formal. But in this context children are accustomed not to ask for the meaning, but to concentrate on a correct application of rules. The form of round games (dialog with partners) and competition games effects that the above mentioned mechanism of correctness-testing is employed. A strong reaction to the question of correctness of the detailed action seems to be especially important. Many pupils don't employ this testing-algorithm when the perform symbolic operations mathematical. One could say ironically: because of a wrong term-transformation nobody screams, nobody protests, nobody smiles: it is just written down. Then there are some pupils, one could call them mathematicians, who perceive that, in the field of mathematics also, there is a correct scope for each rule, that the question whether an operation is applied correctly is a permissible and important question (I don't mean that they can understand the correctness in call cases).

4. Computational Mathematics as an aid to correctness-testing

The meaning of reinforcement is known in experiments on an animal as operant-conditioning. I won't propose now that children are to be reinforced with sweets for every correct term-transformation. But possibilities are indicated, how far incorrect problem-solving have a stronger effect on pupils than written mathematics. Here, a help could be the treatment with algorithms executed by a machine. A machine is marked by fact that it always executes given instructions correctly. Mistakes which occur therefore, have their reason in incorrect instructions. An especial lesson-method might be to use a computer to execute those instructions (rules) which had been found out by the pupil, or to make the development of algorithms, which are executed in dialog with a machine, the subject of the lessons.

An example will be shown briefly - the concept of "computational geometry" developed by Papert at MIT. In the centre of a serie of experiments stands the "turtle", equipped with a pen, as an electric driven vehicle, which can execute geometric constructions according to programs. The programs consist exclusively of simple orders to advance a multiple of a standard distance, angle-divergences of a given degree, raising and lowering of the pen. In the sense of Papert, for example, the comprehension of the geometric concept "triangle" is replaced by the ability to write a program for the turtle, in such a way that it draws a triangle on the paper. An incomplete or incorrect comprehension of the concept leads to an incorrect program, and therefore necessarily to an incorrect figure. The children will experience the wrong drawing of turtle directly, as a consequence of their incorrect orders. A dialog-situation with the computer is established in a particulary clear manner here. A second example is the construction of a programm by which the turtle draws a circle, according to an approximation by a polygon. Children can be led independently to basic ideas of the Archimedes'-method, quite early in their lives. They discover connexions of angles in the regular polygon. This example demonstrates how the computer can be used as medium of verification of hypothesis. For a pupil it is a great help in problem-solving if the veracity or falseness of a hypothesis is not only fixed but obtrudes in his experience. This produces a stronger motivation to continue the trial problem-solving. Arising from this work with children, Papert invented the term "computational geometry". He doesn't mean computer-geometry, which means geometry which is produced on or with the computer. The most important thing here is not the computer as a calculating or drawing machine, but is an analysis of geometric concepts, with the aim of replacing these terms by a sequence of real executable methods. Here we see links with a philosophy of the mathematics of computability. It means here that all concept-formations are connected with methods of production of approbriate objects.

My own investigations (Cohors-Fresenborg 1978) have verified the meaning of a medium for verification of hypothesis in problem solving. In these experiments, for example, children had to construct networks of automata with which one can count to three periodically. The networks of automata used were nets, which worked sequentially, where the flow of information was presented by a pencil being pushed through the net by the pupil. A net which doesn't yet provide the desired solution can be analysed sequentially, just as with the turtle on its wrong path. Moreover a mistake in the operation of such a network of automata must be taken note of in a very different way from a mistake in an argument set out on paper.

After these reflexions we will come back once more to the type of tasks which we introduced on page 4. This task can be given to the pupils on two levels: first the pupils have to build the net with bricks (which can only be joined together in a certain way). In this formulation of the task we seldom find syntax-mistakes in the accomplished solutions, because the material prevents most such mistakes. The result is: many incomplete solutions. Secondly, the children have to solve such problems graphically. Here almost all solutions are "complete", but many are syntactically incorrect solutions which can't be realized with the box of bricks.

In the graphic formulation of the task one can't separate mistakes due to carelessness or inattention from mistakes caused by lack of understanding, but in the building with bricks the inability is obvious.

An apparatus (in the widest sense) stimulates and creates a dialog-situation which doesn't exist for many pupils in written taken for granted mathematics. In this connexion I remind you of the fascination which comes from such a dialog when someone first uses an interactive computer-system. The computer gives us the possibility here of solving problems with an acting-context. It fills the gap

which exists between the experience of incorrect application of rules in a game of the school-yard and an incorrect term-transformation in the pupil's note-book, which has no direct consequences. The computer replaces in certain respects the conditioning to correctness-testing to which mathematicians are educated in course of time. In this context we can state that "pottering" with technical apparatus also creates such a dialog-situation: an incorrect working apparatus presents a clear reply, a sequential analysis of mistakes can often be deducted from incorrect functioning. In this context we also have to mention a research by Lowenthal (1978) who used games on graphs in order to introduce logical analysis of rule-connexions in trains of action for pupils of the elementary school.
The meaning of mathematics orientated in action can be summarized in the sense of Apostel (1978 pp. 247-249) as follows: "thinking is a form of internal development of acting patterns in order to improve efficiency and width of the actions of the thinker."
In occupation with games and automata "children learn practical acting-theory and so this acting-theory should be the basis of a philosophy of mathematics."

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THE PERRY DEVELOPMENT SCHEME AND THE TEACHING OF MATHEMATICS

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Professor: In the logical structure, one can define a function the way we just did, using ordered pairs. Or the term could be taken as undefined, as we did with terms like 'set.'

Student A: Which way is right?

Student B: Are there other ways of fitting the concept into a logical structure?

Student C: Which of these ways has more meaning for me?

The past eighty years have seen a dramatic revolution in the academic conception of knowledge. University examinations set at the turn of the century consisted primarily of questions requesting recitation of facts, whereas most questions on more recent examinations require comparison of two or more concepts. (Perry, 1970, p. 4) Only recently has it been considered appropriate to teach more than one structure or theory.

But is it considered appropriate to do so in mathematics? The scenario above, from a mathematics classroom, is as rare as it is recent. English teachers are notorious for presenting a variety of disagreeing interpretations of a poem, and certainly philosophy professors expound on differing theories. Mathematics jargon, however, is full of certainty; we have 'right answers,' 'truths (theorems)', and 'proofs.' Indeed, more than one student has said to me, 'I went into mathematics because I wanted to have definite answers.'

In the 1950's a counselor at Harvard named William G. Perry, Jr. undertook a study to delineate the 'personality traits' accompanying various conceptions of knowledge. The longitudinality of his research revealed that students' epistemologies actually changed drastically over the years, and Perry abandoned the 'personality trait' movement in favor of a more developmental approach. In 1970 he published his results: a statistically-defensible scheme of nine 'positions' from which various students successively viewed their worlds. (Perry, 1968, 1970)

In this paper I shall critically examine the scheme and research on the scheme. Then I shall focus on applying the scheme to mathematics and its teaching.

I. The 'Perry Scheme'

Perry presents the nine positions on the scheme in the order in which the prototype student would 'take them on.' (Position 1 actually is an extrapolation from the data, as none of Perry's subjects actually saw things this way at the time he interviewed them.) Perry points out that the scheme differs from traditional 'stage theory' developmental schemes in that there is no uniformity in duration of the various positions, and a person might have a range of positions at the same time (1970, p. 48). Also, some persons did not move through positions in the usual order, but 'retreated' to earlier positions or 'escaped' from the development approximately in the middle.

Position 1, called 'Basic duality,' is held by the extremely naive person who has been sheltered all his life from diversity. To such a person, the opinions and lifestyles of authorities and of persons close to him are right, and all others are wrong. Moreover, 'right' is inseparable from 'good,' as are moral wrong and epistemological wrong, so the cognitive and the ethical are closely linked. To a person in this position, such as Student A in the scenario above,

> Right Answers for everything exist in the Absolute, known to Authority whose role is to mediate (teach) them. Knowledge and goodness are perceived as quantitative accretions of discrete rightness to be collected by hardwork and obedience (including the requirement to read all assigned books word by word from the beginning). (Perry, 1979)

I personally identify Position 1 with my thinking (while very young, of course) that a neighbor was wrong (and bad) because she used a different laundry detergent from the one my family used. Perry quotes from an interview with a freshman who remembers entering college with this position:

> S. Well, I come, I came here from a small town. Midwest, where, well, ah, everyone believed the same things. Everyone's Methodist and everyone's Republican. So, ah, there just wasn't any . . . well, that's not quite true . . . there are some Catholics, two families, and I guess they, I heard they were Democrats, but they weren't really, didn't seem to be in town really, I guess. They live over the railroad there and they go to church in the next town. (Perry, 1979)

This 'Garden of Eden' does not survive amidst the diversity of higher education. Even if, by chance, one's parents and school teachers did not openly disagree with each other, one's professors are not likely to present a unified front of certainty. Here they are, the epitome of Authority, and they can't answer a question directly. Two of them might even have different answers to the same question! Position 2 students acknowledge the existence of a multiplicity of opinion, but do not acknowledge its legitimacy. At worst, some of those Authorities are shams. At the very best, the Authorities are trying to teach students how to find the Right Answers for themselves.

> S. Yeah, it wasn't in the book. And that's what confused me a lot. Now I know it isn't in the book for a purpose. We're supposed to think about it and come up with the answer! (Perry, 1979)

The amount of conflict in a student's experience increases, to the point that a new worldview emerges. 'Ah. Maybe I will have to be the one to find the Answers,' says the student. 'Perhaps the Answers really haven't been found by the Authorities--yet.' A student at Position 3 recognizes multiplicity as legitimate in some areas, but assumes that it is temporary. As Perry puts it, 'The Truth is still to be found in the La Placean universe.

> S. I'd feel (laughs) rather insecure about these philosophical things all the time and not coming up with any definite answers. And definite answers are, well, they, they're sort of my foundation point. In physics you get definite answers to a point. Beyond that point you know there are definite answers, but you can't reach them. (Perry, 1979)

But then what is it that Authorities in those fuzzy areas use for evaluation--of students, of colleagues, of theories? Do they use styles of presentation? glibness? personality? To a student in Position 3, the only fair basis for evaluation is the amount of work done. 'I deserve a good grade because I spent three hours on this problem.'

At this point I must digress to prevent some defensiveness. It should be realised that each position encompasses much of the previous one. Thus the fact that we may feel guilty when we don't know something does not mean that we're at Position 1. Nor are we necessarily at Position 3 when we take into account a student's effort when assigning grades. (Of course, such consideration might affect the student's own conception of knowledge.)

The Position 3 student need not remain long in the university before he finds that evaluation is not a function of the amount of effort expended. His friend, or even he himself, tossed off a quick paper that received a good mark, while long hours of effort frequently resulted in poor grades. From that realisation the student moves into Position 4 in one of two ways:

a) 'Multiplicity correlate.' The world is divided, still dualistically, between those areas in which there are Right Answers, and those many areas in which everyone has a right to his own opinion. In the latter areas it is unfair for Authorities to make judgments on any basis whatsoever.

b) 'Relativism subordinate.' What Authorities really want students to do is to develop logically coherent structures that are consistent with the data.

Position 4b differs from Relativism (Position 5) in that relativistic thinking is in the domain of 'the way the world is' rather than 'what Authorities want.' The transition to Position 5 is merely a switch of figure and ground. On the other hand, transition from 4a to 5 must include a realisation that not every opinion is equally valid; there are some opinions to which nobody has a right, since they're inconsistent with the data, or they're not logically coherent.

Whatever the path to Position 5, the relativistic conception of knowledge at last makes the student comfortable with his academic surroundings. That's what the professors have been doing all along: examining different contexts, pursuing implications consistent with various structures, thinking about thought itself. And the student, such as Student B at the beginning of this paper, can do that, too, without depending on Authority for Truth. For 'Truth' has become 'truth,' relative to context.

> Some interpretations will then be found 'better,' others 'worse,' many worthless. Then after all the best analyses are made, there remains, and foreseeably will remain, areas of great concern in which reasonable people will reasonably disagree. It is, in this sense, that relativism is inescapable and forms the epistemological context of all further developments. (Perry, 1979)

But what now? The student is cognitively comfortable again, and there's nowhere to go. Or is there? What implications does a relativistic conception of knowledge have for one's life? As a non-trivial example, what meaning does life have? (There's nobody to tell me!) What career should I choose? (Is there not one that's right, or even best for me?) How do I choose a mate? (What makes one person 'better than' another?) Faced with what appears to be an existential abyss, some Position 5 students either 'retreat' to dualism or 'escape' to cynicism.

Most, however, press onward.

> 5. It took me quite a while to figure out that if I was going for something to believe in, it had to come from within me. (Perry, 1979)

The Position 6 student realizes that to make order out of the chaos he himself must make some commitment--to a career, for example, or to a spouse. This is not the same commitment that might have been inherited from his adolescence, during which he decided he was going to be a physician 'to help people.' It's a 'Commitment' arising from the turmoil of the Position 5 identity crisis. 'To find my identity, I must decide--for myself--what I am going to do, what I am going to believe in.' At the same time there often is a fear of narrowing, of the road not taken.

So the developmental emphasis turns from cognitive to identity. Position 7 students actually make the initial plunge into Commitment, the first leap of faith, frequently with all the enthusiasm of a new religious convert. But then what? There are many more decisions to be made other than career and spouse. What about life style? religious activity? relationships with family? dealings with other people? having children? The list goes on and on. Persons viewing life from Position 8 have come to realize that an indefinite number of decisions must be made. ('And I thought I was having to narrow!') Moreover, if one's identity is to be found only at the end of all those Commitments, then it never will be discovered. No, one's identity is in the ways one goes about making Commitments--in the values one brings to bear in the process of making those decisions.

It never stops, really. After all, the system must be responsive to the increasing amounts of data. Values change. The hypotheses and values accepted on faith must be tinkered with, revamped, discarded. This process will continue throughout one's life. A person at Position 9, apparently, is comfortable with that realization.

One very important point should be made before moving on to examine further research on the scheme. It should be noted that movement from one position to another through the scheme is frequently accompanied by a great deal of grief, of mourning the loss of one's former self. (Perry, 1978) The existence of Right Answers and Authorities was a very comfortable world, at least in retrospect. The freedom of 'everyone has a right to his own opinion' is a pleasant memory to one with the responsibilities of relativistic thinking, and certainly to one who is being torn by a Position 5 crisis. Life in the Position 8 search is quite lonely, and the enthusiasm of the initial commitment is missed. A developmental scheme is quite sterile in contrast to the reality it is trying to describe!

II. Criticisms and further research

One of the first criticisms that comes to the mind of a person first encountering Perry's scheme is often that he is being dualistic about relativism. Is he not espousing the 'higher positions' as an absolute truth? A little thought clears up the matter easily; Perry is not claiming that the later positions are 'better' than the earlier ones, but rather is describing the development he found in his subjects. Moreover, the fact that this development took place in a university setting is consistent with the contemporary academic conception of knowledge as relativistic, as indicated by examination questions.

The criticism is more applicable, however, to later theorists. For example, Knefelkamp (1975) clearly states that the scheme should be considered as a model of liberal education. She thus changes the purpose of the scheme from descriptive to prescriptive, and has convinced Perry personally that such a switch is not inappropriate. (Perry, 1979)

In taking the scheme as a statement of a value system, however, Knefelkamp has not assumed a particularly unorthodox position with respect to the philosophy of education. The ideal of finding identity through relativistic Commitment can be traced through John Dewey all the way back to Socrates' plea for 'examined lives.' That philosophical thread is open to criticism, of course, and a more exact study of the scheme's philosophical underpinnings could be made. Nevertheless, Knefelkamp's adaptation of the scheme need not be seen as dualistic; it could well be Commitment within relativism.

A second criticism of the scheme is its narrowness. It describes a mostly-male population from a selective American university. The interviewing was done by a relatively small group of persons who worked closely together all the time, during a short period of time (1954-63). Does it have any relevance to other kinds of people at other times?

The only major attempt to replicate the scheme in a different context is currently in progress at Wellesley College. In a longitudinal study, Clinchy and Zimmerman (1975) have been interviewing women students and have trained some other students as raters. There are only two interviewers, and the interview method differs from Perry's. These researchers are seeing some differences between their sample and Perry's, and on the basis of those differences are making some refinements in the scheme. Another longitudinal study has just begun at Simon's Rock Early College in Massachusetts (Goldberger et al, 1978) and King (1978) cites one in progress at Iowa State University.

Research by Kurfiss (1977) confirmed the sequentiality of the scheme. Clinchy, Lief, and Young (1977) compared students at two different kinds of secondary schools, Strange (1978) compared college students of traditional and more advanced ages, and several other small projects are underway. Thus there has been little research comparing different populations with the same interviewing and rating methods. As researchers throughout the United States have interviewed and rated students, they have used many different techniques. No common rating manual or raters exist, although attempts currently are in progress to coordinate efforts.

Are interviews even the best way to gather the data? They depend so much on rapport between the interviewer and the subject! Would it not be better to devise an objective test to determine in a uniform way exactly where the students are on the scheme? This third criticism of Perry's research is based on the inefficiency of interviews. Who has the time and money to give long interviews with students, to transcribe the tapes, and to rate them? Unfortunately, alternatives have not produced the necessary data. Back in the 'personality trait' days, Perry began his study with a pencil-and-paper 'Checklist of Educational Values' that was distributed to potential subjects. When he interviewed them later, however, he found no correlation between their answers on the checklist and their positions on the scheme (Perry, 1976). The scheme, after all, describes the forms of the subjects' positions, not the content. Written questions, as well as structured interviews, impose a form on the responses, rather than revealing the student's own structures.

Despite Perry's own reluctance to pursue research with written measures, others have tried. Efforts at the University of Wisconsin at Green Bay proved fruitless (Perry, 1979). A less objective, essay-based measure designed by Knefelkamp and Widick gave better results, with a correlation of .78 with interviews (Knefelkamp, 1974). The gain in efficiency, however, seems to be more than offset by an apparent loss of accuracy in rating, at least in my experience. The trend seems to be toward somewhat structured interviews, now being used at Harvard by Gilligan and Murphy (1978) and at Simon's Rock (Goldberger et al, 1978).

A fourth criticism of the scheme is raised by King (1977), apparently based on a paper by Boyd (1972), and is similar to that of the third criticism. Because the theory address students' 'worldview' rather than more specific attributes, its 'underlying constructs' are vague. There is, of course, no question that this particular developmental theory is broader than those focusing only on intellectual development, on moral development, on ego development, etc. But, as King (1977) also points out, the very strength of the scheme lies in its encompassing of both intellectual and personal development.

To some extent this criticism could be based on a belief that a developmental scheme should be such that a given person at a given time is at exactly one point on the scale. That rather dualistic interpretation neglects to realize that the scheme, like any other metaphor, is a generalization that fits no particular person exactly. As Perry has put it, 'People are bigger than any boxes into which you try to put them' (1976). At any given time, a person may be described as being at several different positions. Perry's instructions to the first group of raters were to imagine the scheme as drawn on transparent plastic and to hold it up against the interview transcript to see where it fits 'least worst.' (Perry, 1976)

This leads to a fifth criticism of the scheme: that a person might be described as being in several different positions at the same time. Perry acknowledges that a person might be at different positions with respect to academics, religion, social life, etc. (1970, p. 48) Kurfiss (1977) found a variety of positions in her subjects with respect to major areas of their lives. Knefelkamp and Slepitza (1976) developed a model adapting the Perry scheme to career development, Meyer (1975) devised a similar scheme to describe religious development, and Gilligan and Murphy (1978) created a model of moral development. But can a person hold different conceptions of knowledge with respect to different academic areas? Could a person consider English, say, to be relativistic but mathematics to be dualistic? Informal observations by Meyerson (1977) and by Buerk (1979) indicate that this phenomenon is possible. If so, how diversified can a person be? Can one view number theory differently from geometry? Can one theorem be taken as absolute and another as relative? If so, does the notion of 'position' make any sense? Research on these questions is just now being undertaken.

A sixth criticism of the scheme is that it is not specific enough in describing identity and ethical development in early positions and in describing intellectual development within Commitment. Consistent with King's criticism cited earlier, Kitchner and King (1979) focused on the single area of intellectual development. Attempting to describe intellectual development beyond Piaget's 'formal operations' as well as characteristics of later Perry positions, they developed a series of stages measured by the development of 'reflective judgment.' At the highest of their stages a person makes judgments based on a synthesis of various viewpoints and with the understanding that different judgments might be equally valid. Unfortunately, the King-Kitchner scale is not at this point particularly useful in understanding the Perry scheme because of the lack of coherence between the two theories in stages after 4, as well as because of the narrowness of 'reflective judgment' compared with the breadth of 'intellectual development.'

A broader attempt has been made to define post-relativistic cognition, this one in terms of dialectics. Basseches (1978) delineated twenty-four characteristics of 'dialectical thought' characteristic of higher-position subjects. His theory is consistent with a view of the entire scheme as dialectic, postulated by Griffith and Heffernan (1978), and with a reiteration by Copes (1978) that post-relativistic conceptions of knowledge can be considered the 'personal knowledge' described by Polanyi (and exemplified by Student C in the initial scenario). In this view a person holds three different conceptions of knowledge as he moves through the scheme: an absolute conception, a relativistic conception, and a notion that knowledge is personal and dynamic.

Another limitation of Perry's research is that his data give only accidental glimpses of formal academic experiences that are conducive to change in persons at various positions. Certainly the overall diversity of a university environment gives impetus to multiplicity; what, if anything, leads to relativism, or Commitment? Pressure to decide on a career or college concentration may not necessitate a move into Commitment, for decisions may be based on little or no 'soul searching.' Does sending a sociology or history paper back repeatedly for reworking encourage movement along the scheme? What can one do in other areas?

The beginnings of research in this area are appearing slowly. Widick, Knefelkamp, and Parker (1975) report on an intervention study at the University of Minnesota with a course called 'Themes in human identity.' The instructors designed course strategies to provide the different kinds of support and challenges appropriate for dualists and for relativists. By the end of the semester, a large percentage of dualists had become multiplists and some relativists had moved toward Commitment. A somewhat later intervention by Stephenson and Hunt (1975) with the same course had similar results. At the University of Buffalo, Meyerson (1977) experimented with intervention in a methods course for secondary mathematics teachers. This work built on research by Copes (1974), who examined the theoretical consonance between various teaching models and development in the Perry sense.

A final area of research covering the scheme relates to its connection with other developmental theories. If the first five positions can be seen as tracing the development of formal operational thinking, as described by Piaget, then the intellectual development during the last five would give clues about the natural of post-formal operations, as Basseches (1978) claims. The first half of the scheme also is similar structurally to the model of Harvey, Hunt and Schroeder (1961), especially as reported by Hunt (1966), although that other model does not address the mesh between identity development and cognitive growth. The theory of ego development proposed by Loevinger (1976) was not addressed directly by Perry, but researchers at Simon's Rock Early College found a correlation of .59 between positions on the two schemes. Another major developmental scheme is that of Kohlberg, describing moral development. (See Kohlberg and Gilligan, 1971.) One of the more recent pieces of new research on the Perry scheme relates to its connection with Kohlberg's model. Gilligan (1977) found that use of the Perry metaphor enabled understanding of what appeared to be anomalies in the movement of women along the Kohlberg scale, and has developed an interview format to rate both Kohlberg and Perry positions. (Gilligan and Murphy, 1978)

One of the most frequent philosophical criticisms of the scheme will be the subject of the rest of the paper. That is that the scheme seems irrelevant to teaching science, especially mathematics. (See, for example, Entwistle and Hounsell, 1975.) After all, there are right answers in mathematics, aren't there? What could it mean to consider mathematics as relativistic, or dynamic? As a student said to me recently,

> I declared a math major because I was upset with the lack of answers in other areas. I mean, in English there are questions you just can't answer. I thought that all math questions had answers, so I decided to go into math.

Lives of students like this one could be profoundly affected by our answers to this criticism.

III. Conceptions of mathematical knowledge

In the earlier discussion of the Perry scheme, I outlined three conceptions of knowledge reflected by positions on the scheme: absolutism, relativism, and dynamism. In proposing that structure, however, I now believe I fell victim to a trap similar to one in which King may have fallen with her 'reflective judgment' notion; that is, I didn't distinguish sufficiently between multiplism and relativism. Therefore I wish to expand my structure to include four different conceptions of knowledge: absolutism, multiplism, relativism, and personalism (or dynamism). Now, how do these structures apply to mathematics?

The best way I have found to answer that question is through an historical approach. I must preface such an outline with a disclaimer, however. I certainly am not an historian, of mathematics or anything else; hence much of what I say may seem somewhat superficial and perhaps even naive. Please consider my model not as a perfect, detailed scheme describing the development of mathematical thought, but rather as a metaphor illustrating the four conceptions of mathematical knowledge. I believe that a case can be made for the emphasis of different conceptions at different periods of western history, but certainly all four conceptions of mathematical knowledge were present at most times. (Besides the standard works in the history of mathematics, I have found quite helpful for my purpose here an undergraduate text by Campbell, 1976.)

Outlines of the history of mathematics usually begin with the Babylonians and Egyptians, and this one is no exception. As usually described, Babylonian and Egyptian mathematics can be taken to be fairly absolute. Mathematical facts were Truths, although not necessarily very general Truths, and these Truths were used for finding out about the way the physically-observable world was. What were the boundaries of my land before the Nile flooded? How do I calculate the length of the other side of this triangle?

In fact, a rather absolute conception of mathematical knowledge dominated Western thinking right up until the nineteenth century, although mathematical Truths became more generalized with the Greeks. Euclidean geometry was considered 'Absolutely True,' of course, and early ideas in algebra and analysis arose from attempts to discover Truths about the physically-observable world.

Within the expansion of the investigation of these Truths, however, lay the seeds of the collapse of the goal. Elementary algebra was not so much a collection of Truths as a language for describing those Truths--a subtle shift of perspective. The crack was found for the wedge of non-Euclidean geometry in the middle of the nineteenth century. Perhaps mathematics need not be verifyable by observed physical phenomena. Different mathematical systems might contradict each other. To remove any doubt, Cantor's playing with the notion of infinity later in the century took all emphasis away from the physical world. It was natural that Hilbert described mathematics at the end of the century as a game played with symbols on paper. Any legal sequence of marks that started from any 'axioms' whatever and followed the logical rules was valid mathematics. Finally, hopes for proving the logical consistency of those axiomatic systems, for for completing one of them, were dashed by Godel. The conception of mathematical knowledge was the multiplistic 'everyone has a right to his own axiom system.'

To some extent, the philosophy of mathematics has made little progress beyond this point, except in acknowledging that mathematicians somehow seem to find meaning in a discipline that has been proved to be meaningless. (See, for example, Goodman, 1979.) They always are searching for 'better' proofs of particular theorems, which means more aesthetically pleasing proofs. (See Halmos, 1968.) And there really are some standards--such as following clearly-stated logical rules. 'Nobody has a right' to trisect angles with only a straightedge and compass, or to generate those awful attempts at Fermat's Last Theorem. But then again, different approaches might be equally valid; sometimes it is better to look at matrices as elements of a ring and sometimes it's preferable to consider them as linear transformations. It depends on the context. In some contexts 4+5=9; sometimes 4+5=11; and at times 4+5=2. Mathematical knowledge really is relativistic.

More recently, I believe evidence of a 'dynamic' conception of mathematical knowledge has been emerging. Mathematicians make strong commitments just as much as other people do. Why choose the logical rules of constructivism over more traditional systems? Why devote a fair portion of one's life to a new approach such as infinitesimals in calculus? Within the context of relativism, mathematicians commit themselves to particular approaches that 'make sense' personally to them, even while realizing that many equally-valid choices are possible. They study the same mathematical ideas repeatedly throughout their lives, finding new meaning and deeper understanding in an idea each time they return to it. Mathematician David Henderson of Cornell University writes

> Instead of asking of a mathematical statement, 'Is it true?' suppose we ask 'Can I imagine it (or see it) as true?' Now, first of all, notice that we no longer have the Law of the Excluded Middle (LEM). (The LEM says that a mathematical statement is either true or false.) If you give me a mathematical statement I might imagine (see) it as true or I might imagine (see) it as not true or (a third or middle possibility) I could not be able to imagine it as either true or not true. Some examples: I can see that 2+2=4 is true--I experience some image like transforming · ·   · · into ⁞⁞ . And I can see that 'All triangles are equilateral' is not true. But I can't see or imagine if 'All maps (in the plane) can be colored with four colors so that no two countries which share a common border are colored the same' is true or not true. Someone has recently claimed a proof that the above statement is true. So presumably I'll never be able to see the statement as not true. However, it is certainly different to be able to say 'I can see (imagine) that it is true' as opposed to 'There is a proof that it is true.' (1977)

Thus four conceptions of mathematics can be described. All of them occur frequently in mathematical research--but not in our classrooms. Even assuming that Perry's scheme should be taken as prescriptive, is it possible to communicate the different conceptions of mathematical knowledge?

IV. Applications to teaching

For most of us, teaching includes much more than preparing for and meeting classes. We set examinations, meet with students individually, order books, show films, and so on. I'll address some of these non-classroom activities later in this paper. First, let us consider how classroom experiences relate to the Perry development scheme.

Three major teaching models dominate in college teaching, at least in the United States: lecture, discussion, and PSI (which statnds alternatively for 'Personalized System of Instruction,' 'proctorial system of instruction,' and 'prescriptive system of instruction,' and generally follows a pattern defined by Keller (1968). Of course, many teaching *methods* or *styles* are combinations of these *models*. I wish to examine how each of these models can be used in mathematics courses to communicate various conceptions of mathematical knowledge; extrapolation to various combinations I'll leave to you.

Lecture model

The lecture model is characterized by the instructor's talking and the students' (theoretically) listening and taking notes. It dates back for centuries, of course, and still is associated with an absolute conception of knowledge. The instructor communicates orally 'the facts' to the students, and perhaps even 'professes,' illustrating publically how learning those facts is to take place (a Perry position 2 conception of knowledge).

What would be involved in practicing the lecture method in ways consistent with other conceptions of knowledge? Experience with disciplines other than mathematics can give us clues. English instructors, for example, talk about several different interpretations of a poem or novel. History professors provide a number of different structures for the data. What can mathematics teachers do?

To be consistent with a multiplistic conception of mathematical knowledge, a mathematics teacher would lecture on various 'approaches' to a piece of mathematical content. But what would this involve? In the previous section, different multiplistic approaches were seen as different axiom systems, with a pervading view that, as meaningless marks on paper, they were equivalent. The multiplistic mathematics instructor might lecture on several axiomatic treatments of a particular mathematical idea. Indeed, it could be argued that the 'modern maths' movement, with its stress on structure and logical systems, supported a multiplistic conception of mathematical knowledge. On a more mundane--but perhaps more practical--level, the multiplistic instructor might take pains to demonstrate several algorithms for a calculation, more than one definition of a term, or a number of proofs of a particular theorem. (Although my experience as a student does not include this kind of teaching, it certainly includes the associated pervasive attitude that the mathematics consists of the marks on the chalkboard and need not have any meaning for the persons involved.)

A more relativistic lecture would do more than expound on a number of different algorithms, definitions, proofs, or axiom systems. It would include comparisons and contrasts of the approaches. For example, this division algorithm is more efficient, while this other division algorithm captures the basic principles better. The algebraic proof of the irrationality of $\sqrt{2}$ often is said to be more elegant, but the geometric proof has the advantage of reflecting better the Greek spirit of the theorem.

A lecture consistent with a dynamic conception of knowledge is somewhat more difficult to imagine. The teacher might model thinking about mathematical ideas in a personal way. He might share his own ways of thinking about the ideas, his own methods of gleaning insight from other people's proofs and definitions. He might take particular problems and develop concepts needed to solve them.

Would such teaching really communicate a personal conception of knowledge? Only partially, I believe. For implicit in a person's moving into Commitment is recognizing that making Commitments is necessary in order to find meaning. This goes beyond recognizing that one *can* make Commitments to find meaning, and perhaps even beyond actually *making* Commitments to find meaning. How many students can succeed, as most working mathematicians apparently did, at making meaning by and for themselves? Surely, for most students, more involvement is necessary, along with more guidance from the instructor in the meaning-making activity.

Discussion model

In the discussion model, students respond to each other's ideas, and the teacher serves as a moderator or group leader. This does not mean that the professor has nothing to do; indeed, the process of keeping in touch with all members of the group, and of planting ideas through careful wording of summaries and questions, can be quite taxing.

It would be difficult to lead a discussion in a completely dualistic manner, since by the very nature of discussion a variety of points of view are expressed. It is conceivable, however, that a professor could focus on finding The Answers to all questions, and encourage students to participate only if they merely recall facts from an assigned text. 'John, what do you think the answer is?' 'Jane, do you agree?' The discussion model is more naturally multiplistic. Discussion seems to be based on the premise that 'everyone has a right to his own opinion,' especially if the participants don't ever directly contradict each other. Is it not frequently the case that participants in a discussion 'talk past' each other, ignoring differences and not commenting on indefensible opinions?

A good discussion leader can base a discussion on a relativistic conception of knowledge, however, even while being polite. For example, he might pose a problem and ask the class to 'brainstorm' about different approaches, listing them all on the chalkboard. Then, later, he can lead a critical examination of the ideas, with less risk of defensiveness from egos hurt by criticism of pet ideas. He can stress the judgment of ideas in contexts. Why is this proof better than that one? Under what conditions is the factoring method of solving quadratic equations better than using the quadratic formula?

In a discussion based on a dynamic conception of knowledge, students are not just observing a model of personal meaning-making performing in front of them. Rather, they are engaged in that process themselves, under the guidance of the more experienced investigator. 'What reasons do you have for structuring the proof as you did?' 'Nancy, you seem to be envisioning the notion of a function as a graph, while Dorothy, you appear to rely on an ordered pair perspective. Steve, what do you prefer?' The group as a whole is making meaning, and the students are getting practice at an art that they can then apply to their own personal learning.

The discussion model has several drawbacks. One is that students should have some outside source of data and opinion, or else they tend to engage merely in 'sharing ignorance.' Of course, providing such a source never is quite so difficult as getting them to use it effectively. Also, discussions, unless led very well, can be a highly inefficient means of learning. My own experience has showed that they need not be less efficient than lecture-based methods; they may be slower in the first few weeks of a course, while student learning is focused on skills of meaning-making, but the mathematical pace can pick up enough as the course progresses to compensate. Another limitation of discussion-based methods is that they do not lend themselves to communicating an absolute conception of knowledge. The main disadvantage of the discussion model, however, is that its success depends heavily on the personality of the teacher, who must be skilled at sensing other people's needs and cognitive structures.

PSI model

The PSI model is characterized by behavioral objectives clearly stated in advance of instruction, and by a hierarchical set of quizzes that students must pass in sequential order. It is advocated as enabling learning at the student's own pace, although it usually is practiced within the limitations of school terms. It can be organized so as to satisfy the cost accountants, since much of the paperwork and interaction with students is handled by student assistants. Because of the model's heavy dependence on written material and examinations, much of the discussion that applies here also is relevant to other, non-classroom aspects of teaching.

It is quite easy to envision how a PSI course could be designed to communicate an absolute conception of knowledge. Every question can be carefully phrased so as to have a unique answer (as far as the written material is concerned), and the student proctors can be naive about other possibly valid answers. The goal of the student in the course is to acquire as many points as possible, representing exact knowledge of Truth.

Imagining a more multiplistic PSI course is not especially difficult, either. A sample behavioral objective would be, 'The student can describe, with at least 80% accuracy, the five major similarities and four primary differences between Purcell's approach to integration and that of Stein.' Multiplicity would be embedded in a dualistic structure, which could be an important advantage of this model in helping students change their position from dualistic to multiplistic.

It is less clear how a PSI course could be taught from a relativistic stance. Objectives for teaching 'on the other hand' thinking would have to be rather vague. 'The student will be able to discuss the relative advantages of topologies and of metrics.' It is somewhat difficult to determine whether or not such an essay is 80% acceptable. The PSI model seems to be completely inconsistent with a dynamic conception of knowledge. The process of meaning-making requires at least a model, and written material is not especially conducive to modeling any process. 'The student will be able to describe the image that the term "semigroup" brings to his or her mind, and can explain why images that don't come to mind are equally valid' just doesn't ring true as a measurable objective! People get their images from people with more experience, and the PSI method saves money by minimizing contacts between students and faculty members in the process of image-making.

Evaluation of models

So which model is most conducive to growth on the Perry scheme? I'm not going to be absolute in my answer to that question, especially since there are many other possible models that are not so commonly used in teaching college mathematics. (See Copes, 1974, for a discussion of some of them.)

Each of the three models discussed here has strengths and weaknesses for conveying different conceptions of knowledge. The lecture and PSI models probably are best for communicating absolute or multiplistic conceptions of knowledge, and the lecture model might be stretched to relativism. The discussion model seems better for teaching relativism and dynamic conceptions of knowledge, but might be stretchable to multiplicity.

But why not then strive to teach only using discussion, communicating relativistic and dynamic conceptions of mathematical knowledge? After all, isn't our goal to get students as far along the Perry scheme as possible? Perhaps that is a valid goal; but is that strategy the most effective for encouraging growth? Kohlberg and Blatt (1972) showed evidence that the environment most conducive to a person's development in Kohlberg's terms is structured one stage above where he is currently. Widick, Knefelkamp, and Parker (1975) designed the 'Themes of human identity' course around that 'plus one' phenomenon, as they planned different structures for dualists and for relativists. So a major question in determining what teaching model to use for a particular course is, 'Where are the students on the scheme?' Assuming that the answer to that question is typical--i.e., "They're all over!'--then an alternative question is 'Which model, or combination of models, allows the most flexibility to provide support and challenge to individual students?' Can a model or teaching strategy give support to students just moving into a new way of seeing the world--say, as a multiplist--even while challenging those who are overly comfortable as multipliists?

The lecture and PSI models seem to be severely limited in this respect. Even granting that one's positional structure determines, to a large extent, what it is that one perceives in his environment, can PSI materials or a lecture be designed to support and challenge different people in the same position? Perhaps so, if the lecture or materials meet the standards of good literature, and if the lecture presentation qualifies as great acting. How many of us can claim to be able to produce quality like that?

The discussion model seems to lend itself to much more flexibility than lectures. Standard techniques of group process help the teacher as discussion leader respond to individuals as he perceives their needs. To some extent the moderator even can influence the group to respond appropriately to individuals within it. To maintain that flexibility, however, the group must be reasonably small, and the teacher must have a very special personality. How many among us can claim to be that skilled at handling group dynamics and that sensitive to the intricacies of human interactions?

As I write this it strikes me how difficult it is for one to teach well, especially if one's students are not already in love with learning. To find a balance among a variety of teaching models that both matches the personality of the instructor and also is successful with the particular group of students at the particular time in history would seem virtually impossible. It is amazing that it happens at all.

Before turning to describe one of the times the 'virtually impossible' happened to me, I should discuss some non-classroom teaching activities and their relation to conceptions of mathematical knowledge. I'll consider only two such activities, and leave the others for you to think about.

One of the most difficult tasks for me is creating examinations. A question that bothers me incessantly is, 'What kinds of examination questions in mathematics reflect multiplistic, relativistic, and dynamic conceptions of knowledge?' As with similar questions about teaching models, I can turn only to other disciplines and to our earlier consideration of what is meant by these kinds of mathematical knowledge. As a result, I set forth as illustration four different examination questions over the same mathematical idea:

1. Define 'the combination of m things used n at a time (C(m,n)),' and calculate C(5,3).
2. What are two different definitions of C(m,n)?
3. Please list several different definitions of C(m,n). What are the relative advantages of each?
4. What are several different ways in which the concept of 'combinations' can be viewed? What is your personal favorite, and why?

These questions, I believe, are consistent with our four conceptions of knowledge, although I acknowledge that there are not perfectly clear distinctions among them. Unfortunately, if one is, in effect, teaching differently to different students, then it follows that different examination questions are appropriate for different students.

A second non-classroom activity involves meeting with students individually, usually under the pretense of discussing course material. Some of what transpires in these sessions might fall under the rubric of 'counselling,' although I don't mean to refer to the specialized activities for which psychologists and psychiatrists are trained. I mean providing students with appropriate challenges and support as we do in our daily interactions with other human beings.

As I mentioned earlier, people suddenly looking at the world in a way new to them may well experience grief at the loss of their former innocence. Persons facing the Position 5 identity crisis are especially prone to disequilibrium. Frequently a teacher who is seen to be friendly, respectable, and somewhat objective is confronted by a kind of disbelief. 'How do you survive in a world as chaotic as this, where there are no absolutes at all?' As those people entrusted with aiding and abetting liberal education, are we not obligated to help such a student grow in ways other than those directly related to our course? Certainly if we accept Perry's scheme as providing goals for liberal education then we are compelled to respond to such cries. Any of us who are bothered by such 'nonprofessional' activity, however, can find comfort in the fact that the scheme suggests not only an obligation but also an outline of what will help such a student. At the very minimum we can share with the student our own hesitations, our own Commitments within uncertainty, and our own excitement about a dynamic and personal conception of knowledge.

I am reminded at this point of a former, rather dualistic student of mine to whom I shall refer as Jones. Frequently during the semester he would ask a question in class that conveyed a message like 'cut out this "on the other hand" garbage. Stop saying, "If we look at the concept of integral this way we see an area; on the other hand we can also see the integral as a sum," and tell us what we're supposed to know.' The rapport among class members and between Jones and me developed to the point that I could publically tease him--and others who raised similar questions--about the 'Jones syndrome.' About two weeks before the end of the course he walked with me back to my office after class, discussing some assignment or another. After a pause in our conversation he suddenly asked, 'What can I do to get over the Jones syndrome?' I believe that was the most important question any of the students asked during the entire semester.

V. A case study

When I first taught calculus, I taught in a rather typical fashion. I lectured. I told the students what the definitions and theorems and formulas and algorithms were, using (that semester) the Gospel According to St. Thomas. I was careful to match my notation to the text's, to avoid confusing the students, to answer all questions thoroughly, to give difficult--but ultimately fair--examinations, and to write clearly on the blackboard. I was a good teacher--absolutely.

The following year I taught a section of the same course, but I did not present Thomas' Gospel so well. I presented my own Gospel and required that students read Thomas in order to work the assigned homework exercises. I even had them write a paper comparing and contrasting my 'treatment' of the derivative (disease?) with that of the textbook. Numerous other texts were on reserve in the library, and appropriate reading assignments 'forced' students to see alternative axiomatic systems. They even were 'exposed' to some unanswerable questions about calculus--for example, what is the best definition of the Riemann integral? (If they were fortunate, their exposure innoculated them against ever 'getting' a serious case.) I was a multiplistic teacher.

More recently, I have been going further than that when teaching calculus. Class sessions are devoted to developing, through discussion, our own mathematical theory and techniques to solve intriguing, open-ended problems I have posed. Usually one of them is the Monk Problem:

> A monk lives at the bottom of a not-so-high mountain in Tibet. He has a tradition. On the last day of every month, precisely at sunrise, he leaves his hut at the bottom of the mountain and climbs along a certain path to the top of the mountain, timing his trip so that he arrives precisely at sunset. He meditates all night. He leaves the top of the mountain exactly at sunrise the next morning and walks down the same path, arriving back at his hut precisely at sunset.
>
> Question: Is there necessarily a point on the path at which the monk arrives at the same time of day on each of the two days, both going up and coming down the mountain?

In analysing such problems as a group, class members are called upon to contribute their own ideas, to make tentative commitments to particular approaches, and to make aesthetic choices between structurally-equivalent definitions. (Should the maximum value of a function over an interval be defined as it is in the text? Or does it make 'more sense' to define it in such a way that a constant function does not have a maximum value? What are the advantages and disadvantages of each?) When the students write papers contrasting our class approach to the approach of some textbook, they not only see that and how various approaches are different, that and how the class approach is our personal one; they also are asked to decide which approach they personally prefer, and why. Students learn a great deal more from each other than they did before, and I work more than before with individual students as they face crises caused by not finding neatly-wrapped packages of answers at the end of the course rainbow. I have become more of a personal teacher.

The teaching model alone does not create the spark of the 'magical' course, however. A match between the students' personalities and the instructor's is essential. I last taught calculus almost two years ago, and the spark was there. I certainly looked forward eagerly to each class session, and the students seemed to do the same. Students from that class gained a perspective on mathematics, and, I hope, on their own worlds, that they had never experienced before, a perspective that many of them have shared with others while serving as teaching assistants in my later courses. There were dualists who learned that there is more than one way of defining the limit of a function at a point. There were multiplists who figured out that what I wanted was a defense of their opinions. There were even some relativists who got a glimpse of personally-meaningful mathematics. And there were some who didn't change much at all, despite the prodding and joking. I think in particular of the Jones of that class, who hasn't yet reached her crisis and, as of yesterday, was still asking for The Answers in a multiplistic physics course. My hope for her lies in the fact that some students from our calculus class are still kidding her about her perspective.

VI. Conclusions

It is traditional for one to draw conclusions at the end of a paper, to summarize into neat packages what one has said. Then a reader who, like all of us, has no time to wade through all the verbage, can jump to the end and see if the novel is worth reading.

Having endorsed a dynamic conception of knowledge, however, I am somewhat uneasy about drawing conclusions for you. I think that's your job. This paper itself represents my own Commitment within relativism--subject to change, albeit with a great deal of pain. What do you think?

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THE STRUCTURE OF MATHEMATICAL ABILITY

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EARLY STUDIES OF STRUCTURE

Since the early years of the present century a number of research studies have addressed the issue as to whether a group factor of mathematical ability exists over and above 'g', the general factor. These studies, which have attempted to map "mathematical ability" onto one or other of the classificatory systems of human cognitive ability as outlined by psychologists in the psychometric tradition, resulted in a considerable amount of conflicting evidence.

These studies tend to fall into two roughly equal groups. The conclusion of the first of these groups, those who claim that there is not a specific mathematical factor, is exemplified by Hamley's (1935) conclusion that "Mathematical ability is probably a compound of general intelligence, visual imagery, and ability to perceive number and space configurations and to retain such configurations as mental patterns. Each one of these could be subdivided into simpler factors and components". Other representatives of this faction are Cameron (1925), Oldham (1937), Mitchell (1938), Blackwell (1940).

Of the pro "group factor" studies the most thoroughly argued case is represented by Wrigley (1958) who concluded that there was "clear and convincing evidence that the four tests of mathematics form a group factor over and above the general factor, and separate from the number factor". Other studies supporting this two-factor view are those of Barakat (1950,51), Hamza (1951,52), Lee (1955) and Werdelin (1958).

An interesting fact becomes apparent when the publication dates of these two groups of studies is noted. The studies which support the existence of a "group factor" have all been conducted more recently than those which support the notion of 'g' as the principal determinant of mathematical performance. This has inevitably led to a consideration of the effect of the refinement of testing techniques and the advances in the sophistication of the relevant statistical tools over this time interval, with a consequent strengthening of the case for the "group factor" conclusion.

However, Biggs (1962) makes the pertinent observation that the earlier studies were conducted in the older elementary schools prior to the 1944 Act, while the others were mostly carried out in grammar and selected secondary schools, and that a study by Vahervuo (1948) with a primary school sample produced results similar to the earlier work carried out in central or elementary schools. At the primary level, as in the old central or elementary school, many branches of mathematics tend not to be dealt with and mathematics

tends not to be dealt with as a unified discipline. Bearing this in mind and with reference to Wrigley's conclusion, based on his research, that teaching should emphasise the unity of mathematics, Biggs points out that this "seems rather to be putting the cart before the horse, in that it seems that it is only in those schools which emphasise the unity of mathematics that the group mathematical factor does appear. Where mathematics is taught either piecemeal or in one branch only, it is not surprising to find that there is little or no tendency for a special "mathematical ability" to be identified, and that 'g' appears as being of more importance in contributing towards success in these schools than any other single or group factor".

It thus seemed possible that the contentious issue as to whether or not a group mathematical factor existed might well centre on the differences in the way in which mathematics is taught at the different stages of education. However, Biggs, at a time when the importance of intellectual maturation in mathematics education was beginning to be appreciated as a result of a wider appreciation of the importance of Piaget's work, foresaw the possibility of an underlying developmental factor at play. "Thus it may not be so much that there is a group mathematical factor, as that the child's thought processes are developed in a way which happens to be appropriate to a subject like mathematics".

RECENT STUDIES OF STRUCTURE - A CHANGE IN APPROACH

In a recent study of the dimensions of mathematical ability Furneaux and Rees (1976) noted that any systematic attempt to map mathematical ability into a more general classificatory scheme should logically start with an agreed operational definition of mathematical ability. However most of the studies in this area have no more than an implicit definition of ability which is reflected in the selection of tests used to "measure" mathematical ability. Considering the problems of such an approach it is concluded that, "what is surely needed is a dimensional analysis of a data matrix giving information about performance with each one of the large variety of operations subsumed under the term "mathematics", in each of the domains concerned. So far as the psychometrician is concerned, this means that the exploration of the dimensionality of mathematical ability has to be approached via the factorial analysis of inter-item, rather than inter-test correlations".

Furneaux and Rees carried out such an analysis as the item/item correlations within a 50 item test initially devised on the basis of an extended analysis of the performance of Craft students in City and Guilds mathematics examinations (Rees, 1973). This test was administered to samples of 'O'-level secondary pupils, craft and technician college students, college of education students and university undergraduates. A subset of 12 items of this test had previously been noted as representing a group of items which were more difficult for students than teachers expected although they did not represent a homogeneous group in terms of any apparent mathematical process or content area. These items loaded out on the second factor of an orthogonal structure where the first two factors represented reliable group factors (the subsequent factors being of an obviously specific nature and accounting for minimal test variance).

A subsequent study by Furneaux and Rees (1978) involved the administration of a 69 item test to a sample of Craft and Technician students, and the Thurstone PMA battery was also administered. The factor analysis of the inter-item correlations resulting again revealed a two-factor orthogonal structure. In this study a sub-group of items, with characteristics similar to those mentioned above, loaded on the first factor and the remaining items loaded with the PMA scales on the second factor. Examination of the items loading out with the 'g' measures, and study of the solution strategies of students, suggested that such items "present a clearly structured problem which can be solved provided that some standard programme of operations has been learned". This factor therefore appears to represent a "general ability" or 'g' factor.* The test items which loaded on the other factor appear to represent items which require the ability to "conceptualize the problem in such a way that relevant operations can first be identified, and then applied in proper combination and sequence". This factor appears to represent an 'inferential' or 'mathematical ability' factor independent of 'g'.

These studies appear to have succeeded in demonstrating the existence of a mathematical ability factor independent of a general ability factor as a determinant underlying mathematical performance.

In an attempt to trace the existence of a similar 'two-factor' structure at the end of primary education via the dimensional analysis of inter-item correlations Curnyn (1979) administered a mathematics test to 300 pupils at the end of Junior school. The test was constructed bearing in mind the 'g', 'inference' item distinction suggested by the Furneaux, Rees studies, and was administered along with a 'g' type measure AH1 (NFER Publishing Co. Ltd., 1977).

The analysis of this data revealed that at this level, as suggested by the earlier studies, general ability was the main determinant of performance.

Although no mathematical ability, or 'inference', factor independent of 'g' was detected two variable clusters corresponding to the 'g' and 'inference' test-items emerged and were differentiable within the context of an oblique rotation. It seems possible that this data represents the detection of an emergent 'mathematical ability' factor and that the differentiation of 'general ability' and 'mathematical' or 'inferential' ability may be a developmental phenomenon.

The ratio of variance accounted for by the 'g' factor to that accounted for by the 'inference' factor, in these recent studies, may also reflect the developmental nature of the 'inference' factor. In the Furneaux and Rees (1978) study this ratio was 2 : 3 ('g' : 'inference'), the sample being a group of further education students age range 17-19 years.

In an unpublished study by Rees (1978) with a sample of fourth-form secondary pupils age range 14-15 years, this ratio was approximately 2:1, and with the sample of pupils at the end of primary education the ratio is approximately 5:2. Although variation of this type will relate to the exact make-up of the particular tests it is possible that this age-related variation reflects the increasing importance of the 'inferential' factor as mathematics education and/or students' cognitive capacities advance.

It has been suggested that inference type items are those which require the ability to "conceptualize the problem in such a way that the relevant operations can first be identified and then applied in proper combination and sequence". However the children in the primary study sample are, in terms of the Piagetian model of cognitive development, towards the end of the sub-period of concrete operations. Bearing in mind the childs' lack of capacity to generalize and deal with abstract relations at this stage it is not surprising that the performance of the primary sample is much more independent of this mode of reasoning and more dependent on 'g' type ability to the extent that this is represented by items which present "a clearly structured problem which can be solved provided that some standard programme of operations has been learned".

It would appear that at the end of Primary Education a general ability factor is the main determinant of mathematical performance but that there is evidence of the emergence of a mathematical ability factor as mathematics education progresses through secondary to further education. The existence of such a developmental factor may help to explain the disagreements among early researchers as to the existence of a group factor of mathematical ability. If it can be established that the emergence of such a factor parallels the process of cognitive maturation then the technique of factoring inter-item correlations, as well as providing a means of elucidating maths-test structure, may provide a statistical means of studying this process with respect to mathematical performance at the level of the group rather than the individual case.

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* Recent work suggests that this factor may be more fittingly described as a Speed/Accuracy factor, compare Furneaux (1960).

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Conservation of Number: A Factor in Mathematics Instruction with Manipulative Materials

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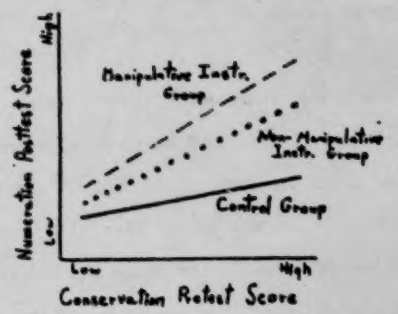
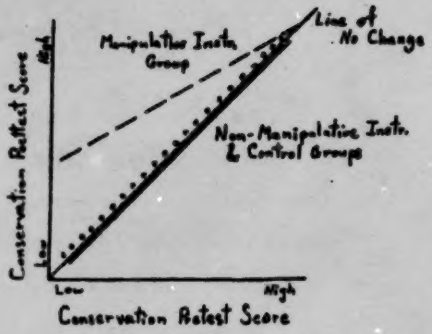
In their review of the literature on activity-based mathematics instruction, Suydam and Higgins (1977) cautiously concluded that studies generally support the use of manipulative materials. However, they were less optimistic than the conclusion might suggest and indicated there was a need for determining the crucial factors leading to the effective use of such materials. This paper briefly describes a research study investigating the impact of one personological variable, conservation of number, on the effectiveness of manipulative materials (Engelhardt & Wiebe, 1979).

Conservation of number (Piaget, 1964) has been thought prerequisite to learning number-related concepts, including numeration concepts (Mpiangu and Gentile, 1975; Souviney, 1975; Copeland, 1978). In manipulating objects like counting sticks or Dienes blocks to learn numeration concepts, conservation of number appears to be important. The child must realize that when ten sticks are bundled, their ten-ness remains; likewise, that the ten-ness associated with trading the Dienes units for a long remains. Thus, one would expect that (1) numeration instruction involving counting sticks and Dienes blocks should have a differential effect upon the learning of conservers and non-conservers -- benefiting conservers more than non-conservers, and, if conservation of number is a critical personological variable for predicting the impact of such manipulative materials, (2) when the effects of conservation of number are accounted for, numeration instruction would be differentially more effective for conservers and non-conservers receiving instruction involving counting sticks and Dienes blocks than for those receiving non-manipulative instruction.

According to Piaget learning is an active process. Our knowledge of the world is actively constructed out of our interaction with the environment. Thus, if a non-conserving child encountered experiences which helped structure the environment appropriately, the child may be led to a higher level of conservation of number. Since structured manipulations with counting sticks and Dienes blocks may so appropriately arrange the environment, one would expect that (3) numeration instruction with those manipulatives should differentially improve learner's conservation of number -- benefiting non-conservers more than conservers.

Couched in regression terms, it was anticipated that (1) the regression lines (numeration achievement scores on pre-instruction conservation scores) for two treatment groups (one receiving manipulative numeration instruction and one receiving non-manipulative numeration instruction) and a control group (no instruction, but free play with manipulatives) would have different slopes -- the difference between treatment group slopes and control group slope increasing with pre-conservation scores, (2) the numeration performance of the manipulative instruction group would generally exceed that of the non-manipulation instruction group -- the higher the pretest conservation score, the greater the difference in performance, and (3) the regression lines of non-manipulative numeration instruction and control groups would have a no-change-in-conservation slope (slope of 1 and intercept at 0), and the regression line of the manipulative instruction group would have a positive-change-in-conservation slope (slope less than 1, intercept greater than 0, and generally above the no-change slope). The illustration shows the hypothesized regression line relations.





Sample. To test these hypotheses a sample was selected, consisting of the entire first-grade population (137 students) in a middle-class neighborhood school in suburban Phoenix, Arizona. Three classrooms were assigned to a manipulative instruction treatment and one class each to non-manipulative instruction treatment and control group.

Procedure. During a two-day period in the early spring of 1979, students were individually administered an eight task test of conservation of number. Each of the five classrooms then received eleven days of (a) manipulative instruction on numeration concepts, (b) non-manipulative instruction on the same numeration concepts, or (c) unstructured manipulative instruction experiences with the manipulatives. All treatment teachers received structured lesson plans and in-service training, and all groups were often observed by one of the experimentors. There were sufficient materials for each child to manipulate. After the eleven days, the students were individually posttested for their numeration achievements (nineteen items) and again for their level of number conservation. All testing was done by a group of trained testers.

Results. The results showed a significant difference among all five classes on initial level of number conservation and between males and females on numeration achievement. Since non-homogeneity of regression of numeration achievement on pre-conservation level was hypothesized, the Johnson-Neyman technique (Ceurvorst, 1978) was used to compare treatment effects, while controlling for the effects of initial level of number conservation. These analyses revealed that for female subjects scoring roughly between 5 and 9 in the initial number conservation measure (transition and conservers), those receiving manipulative instruction scored significantly higher on the numeration measure than those receiving non-manipulative instruction or only unstructured materials manipulation.

Regression analysis for pre-post conservation scores indicated the hypothesized relationships for the control group, non-manipulative instruction group and one manipulative instruction group. Post hoc analysis and discussions revealed that (1) the control and non-manipulative instruction groups had actually received some pre-experiment numeration instruction and (2) except for the one manipulative instruction group, there were few non-conservers in the groups.

Conclusions and Discussion. Several conclusions were drawn. First, for first-grade girls who might be characterized as transition or conservers, numeration instruction involving counting sticks and Dienes blocks appears to be more beneficial than numeration instruction not involving such manipulation (i.e., no numeration instruction). Second, supporting the findings of other researchers, number conservation pretest scores seemed to predict numeration performance, regardless of the type of instruction or non-instruction. Finally, the results do not suggest that numeration instruction involving counting sticks and Dienes blocks differentially improves students' level of number conservation.

The observed sex difference could not be explained, based upon the theoretical orientation of this study. The impact of having few non-conservers in the study, particularly on investigating the effect of manipulative instruction upon level of number conservation, was explored and a replication of the study called for. However, within the limitations of the present study, it does appear that students' level of number conservation is one factor that may determine whether a manipulative approach will, in general, be successful with a given student, at least when numeration mastery is the objective of instruction.

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REVISION OF PROBABILITIES AND THE TIME AXIS

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One of the first topics usually presented in courses of Probability Theory is that of Conditional Probability. Below is a problem that might be illuminating for students of such courses, at whatever level the course is given. It may be given immediately after introducing the concept of "the probability of A given B":

> An urn contains two white balls and two black balls. We shake the urn thoroughly, and blindly draw out two balls, one after the other, without replacement.

Let us denote:

W_I - A white ball on the first draw

W_{II} - A white ball on the second draw

a. Suppose we know that the first ball drawn was white. What is the probability that the second ball is also white?

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In this problem we are seeking the probability $P(W_{II}/W_I)$. Even with no formal training in probability it is easy to see that what follows from the information that the first ball was white is that prior to the second drawing the urn contains two black balls, and only one white ball, so $P(W_{II}/W_I)=1/3$. This, indeed, was the answer given by 62% of 42 naive subjects who were given this problem. The second most popular answer was 1/2, given by 14% of the respondents. On the other hand, let us consider the following question:

b. Suppose we know that the second ball drawn is white. What is the probability that the first ball is also white?

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We ask here about $P(W_I/W_{II})$. If your answer to this question is also 1/3 you are right. The very knowledge that the second drawing resulted in a white ball - even though it occurred after the drawing of the first ball - implies that the possible outcomes of the first drawing were limited to just one white and two black balls, since the self-same ball drawn second could not have also been drawn first. Thus the conditional probability of the first ball being white given that the second ball was white is one third, just like the probability that the second ball was white given that the first one was white. This answer, however, is apparently unintuitive. It was given by a minority of 88 Hebrew University freshmen, taking an elementary course in Probability Theory, who were given questions a. and b. (in that order). Table 1 presents the distribution of responses to question b. given by the 80 students who had answered question a. correctly.

TABLE 1

	$P(W_I/W_{II})$	Number of students	(Percentage)
	1/2	41	(51.2%)
The correct answer	1/3	13	(16.2%)
	1/4	21	(26.2%)
	Other values	5	(6.2%)
	Total	80	(99.8%)

The modal answer, $P(W_I/W_{II})=1/2$, is a mistaken one. What seems to be the origin of this common mistake finds its expression in the following comment, which we quote from one of our students: "The first ball doesn't care whether the second ball was white or not". To which one may reply: "Indeed the ~~second~~ first ball doesn't care, but we do". This exchange points out the difference between the probability of an event as a parameter of the physical situation in which the event occurs (propensity), and the probability of that event as a measure of one's informational uncertainty about its occurrence (degree of belief).

The erroneous answer, $P(W_I/W_{II})=1/2$, is congruent with the view that probability measures some feature of the objective setting at the immediate point of time at which the event takes place. Such an interpretation treats evidence occurring earlier than the judged event differently than evidence occurring later than it, since the former but not the latter may affect the present setting. Thus knowledge of the outcome of the second draw would be perceived as irrelevant to the question about the outcome of the first draw.

When we tried in class to convince students of the formal symmetry between equally diagnostic items of evidence regardless of their place on the time axis, our students often vehemently defended their erroneous answer, $P(W_I/W_{II})=1/2$, by arguing: "At that stage (before performing the first drawing), the second drawing had not yet been carried out, so it could not have affected the probability of W_I". This poses a suitable occasion for the teacher to point out that we are no longer "at that stage". We have advanced beyond it when we acquired the information concerning the outcome of the second draw. Therefore probability, as a measure of uncertainty, should take the later event into account.

A minor change in the structure of problem b. might convince the student who is still skeptical. Suppose the urn contains one white and one black ball. In this more extreme case it is easier to understand that $P(W_I/W_{II})=0$, while $P(W_I)=1/2$. This, indeed, was the reply given by 19 out of 22 naive respondents who were given this version of question b.

The conception of probability as a propensity of the situation would probably be also tied to an interpretation of conditional probability as a matter of cause and effect. The first drawing, being causally independent of the outcome of the second one, would thus retain its original probabilities in spite of knowledge of the second drawing's outcome. Admittedly, one can often utilize causal relationships for the purpose of assessing conditional probabilities, as for example in assessing the probability of rain given the presence of ominous black clouds, but conditional probabilistic reasoning should not be restricted to causal considerations. The outcome of an earlier event is indeed causally independent of a later event, but the *information* about the later outcome might shed new light on the uncertain outcome of the earlier occurrence and may modify our *evaluation* of its probability.

The prevalence of causal schemata in judgments under uncertainty has been studied by Tversky and Kahneman (~~1976~~ in press). Our results are in complete accordance with their thesis that the impact of data on intuitive judgments of probability is not determined solely by their informativeness (as prescribed by the normative theory of evidence), but primarily by considerations of causality. The reason for depending so heavily on causal connections might be, as in our example, the conception of probability as an inherent component of the situation.

In various fields of science one infers from a later event the nature of an earlier happenning. Besides, people do encounter in their daily experience situations where recently obtained information is utilised to adjust prior evaluations of the probability of certain events. The teacher may want to capitalize on the existing familiarity with such practices in convincing his students of the possible relevance of later outcomes for former events. Thus, for example, the findings of archeological excavations throw new light on historical events which took place hundreds or thousands of years earlier. In another context, although diseases are the causes and symptoms are the effects - although diseases come first and symptoms follow - the medical diagnostic process attaches probabilities to diseases on the basis of determination of their symptoms.

Conditional probabilities are a tool which allows us to revise probabilities. For this revision one should consider all the information at one's disposal, regardless of neither the temporal nor the causal relations between the conditioned and the conditioning events.

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Is it Possible to Measure the Intuitive Acceptance of a Mathematical Statement?

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The problem

At the Osnabrück Conference of the IGPME (1978), we have described a number of general characteristics of intuitions: immediacy (self-evidence), coercive influence on the ways of thinking, extrapolative capacity, globality, resistance to teaching and age influences.

Our problem was to find techniques to identify and measure objectively those dimensions and, of course, to determine their interdependence in concrete situations.

Consequently, we started with what we have then considered as being the basic characteristic - intuition, as immediate knowledge. In order to explain what we mean by "immediate knowledge" we wrote: "An intuitively accepted truth is self-evident - no arguments appear to be necessary to convince us that the given statement is true" (See proceedings of the Conference, 1978 p. 149). Trying to find an operational definition of immediacy and to produce an objective instrument for measuring it, we found that "immediacy" consists, in fact of, at least, two different ingredients, irreducible one to the other. The first is the *degree of confidence* the subject has in his one answer. The second is the *degree of obviousness* of the same answer as it appears to the subject.

Let us take some examples. Let us consider a geometrical illusion for instance the famous illusion of Müller-Lyer:

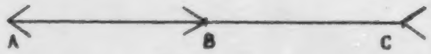


Figure 1

The segment B C *appears* to be longer than the segment A B. As *immediate* knowledge, expressed by its obviousness, *BC is longer than AB*. At the same time I know that it cannot be so, because, when constructing the segments, I took care to draw them equally long. I may have some doubts on the correctness of my drawing, but I am surely not convinced that BC is really longer than AB. We are confronted here with a situation in which, what appears perceptively obvious, contradicts what we can accept as being *really* true. As a matter of fact, (apparent) obviousness is contradicted here by an opposite conviction.

Can we affirm then, that the statement: "BC is longer than AB" is an *intuitively acceptable statment*? Of course not. What is lacking here is the element of conviction, of *intrinsic* conviction.

An intuition is not reducible to a pure perception. An intuition is always an interpretation. As a pure perception, BC *is* longer than AB. On the other hand, the statement "In figure 1, BC is longer than AB" appears to be, at least, as a doubtful one. When considering the Müller-Lyer image, the feeling is of some inner contradiction and not of intuitive acceptance.

We take an opposite example. Let us consider the well known formula for solving quatratic equations.

$$X_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Those who are used to algebraic calculations have no doubts on the validity of the formula. They are absolutely convinced that the formula is correct. In spite of this, *they have not, generally the feeling that the formula is intuitively evident*. The formula lacks obviousness. Consequently, *we may be highly convinced of the truth of a statement without being intuitively satisfied by it*. If, instead of "-b", the formula would contain "+b" my intuitive reaction would not be that of wonder or rejection.

Let us consider a third example.

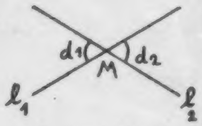


Fig. 2

Two straight lines l_1 and l_2 intersect in point M. The fact that the two opposite angles d_1 and d_2 are equal, appears to us, generally, as being self-evident. On the other hand, we have also no doubt of the angles being equal. As a matter of fact, the equality of the angles appears to me as being intuitively acceptable. No further proof is needed, no further perceptive evidence is requested in order to make sure that the angles are and must be equal.

Therefore we submit that the feeling of direct acceptance which basically characterizes intuitive knowledge is determined by the confluence of two factors: level of confidence and obviousness. It may be supposed that some correlation and some interraction exist between these two components, but, it is not always the case. In order to assess the immediate acceptance of a fact by a subject both aspects have to be taken into account.

The main purpose of the present research has been to investigate the possibility of obtaining more systematic information about the subject's feelings of confidence and obviousness towards his own solution or interpretation.

Method

The questionnaire

Eight mathematical problems presenting various degrees of complexity and apparent paradoxality were used. We started, in fact with eleven problems, but we dropped out 3 of them for some methodological reasons. The 8 problems were divided into two parts, each part constituing a separate questionnaire. A subject had to answer to only one of these questionnaires. The rationale of the division was only the necessity to avoid boredom and tiredness in the subjects.

After solving the problem, the subject had to explain his solution. The next step for each problem consisted of six check questions referring to the intuitive acceptance of the solution by the subject itself .

A. The problems

1. In a plane there is a line l and a point C not on the line. Is it possible to draw, in the same plane, through point C, a second line which will not intersect line l?

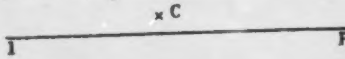


Figure 3

2. On a segment AB there is a point C chosen randomly. Let us divide the segment AB into two equal parts. Let us divide into two equal parts each of the two halves of the segment AB. Let us continue to divide in the same manner.

Shall we arrive at a situation such as that one of the points of division will hit exactly point C?

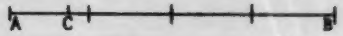


Fig. 4

3. Two lines intersect at point M. Are the angles d_1 and d_2 equal?

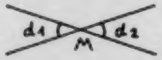


Fig. 5

4. Given a segment AB=1m. let us add to AB a segment BC=½m. Let us continue in the same way adding segments of ¼m, 1/8m, etc. Will this process of adding segments, as described above, come to an end?

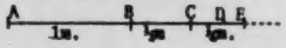


Fig. 6

5. Let us consider again question 4. What will be the sum of the segments AB + BC + CD .....etc?

6. Let us consider the following two sets: the set of points of the segment AB and the points of the square CDEF.

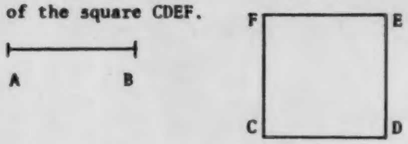


Fig. 7

Is it possible to match to each of the points of the square one and only one of the points of the segment AB? (It is forbidden to use a point of AB more than one time).

7. Let us consider the following two sets:

1) The set of the natural numbers:

N =(1, 2, 3, 4, 5, 6, ...)

2) The set of the points of a line l:

Let us consider the set of points of a line segment and the set of points of a line. Is it possible to match to each point of the line one and only one point of the segment?

Is it possible to match to each of the points of the line l one and only one point of the segment AB? (It is forbidden to use a point more than one time).

The subjects have had to answer to each question by "Yes " or by "No". They have had to explain their solution and to answer to each of the six check questions intended to measure the intuitive acceptance of the solution by the subject itself . For measuring the degree of intuitive acceptance we have used two groups of "check questions". One referring to the level of confidence, the second referring to the obviousness of the solution.

Check questions for the level of confidence:

1. Have you doubts with respect to the correctness of your answer?
2. If a mathematics teacher tells you that your answer is wrong, will this cause you to doubt your answer?
3. Do you feel that a formal proof will increase your confidence in the answer you gave?

Check questions for the degree of obviousness:

1. Do you need some effort in order to completely agree with your answer?
2. Is your answer self-evident for yourself?
3. Do you feel that, through a concrete illustration, your answer will appear to you more obvious?

The subjects had to answer to each question by Yes or No.

In the questionnaires the questions were mixed. The order was the following: 1c, 2c, 3o, 3c, 1o, 2o. The symbol c stand for "confidence" and symbol o stand for "obviousness".

A hierarchy analysis proved that each of the two groups of check questions formed a Guttman scale. For each of the 8 problems the coefficients of reproduceability were higher than 0.9 thus proving that both scales are, in each case, valid ones.

The coefficients of scalability were, in all cases, above 0.6 thus proving that they are cumulative and unidimensional in relation to each of the eight problems. It is in that order that we presented them above i.e. $3 > 2 > 1$ for both groups of questions. The scores assigned to the subjects' answers were weighted by coefficients corresponding to their place in the established hierarchy. The coefficients were 3, 2 and 1. For instance, a subject who would answer "No" to each of the three questions referring to the level of confidence would get 3+2+1=6 points (the highest possible) expressing his high confidence in his answer. Therefore the resulting scores for each of the two dimensions (confidence and obviousness) could range from 0 to 6).

The subjects were pupils from grades 8 and 9 enrolled in different high schools situated in 4 different regions of the country. According to the programes in use for mathematics teaching, the pupils were supposed to have some knowledge with respect to the notion of infinity (for instance the set of natural numbers is infinite, a line is infinite etc); rational and irrational numbers ; the concept of one-to-one correspondence. They were not supposed to know the meaning (and the Cantorian hierarchy) of transfinite numbers; the concept of non Euclidean geometry.

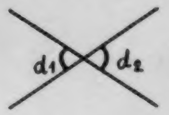
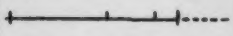
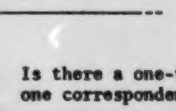
As it can be remarked, all the problems but one (problem 3) deal with the notion of infinity. We have especially referred to it, because infinity has large intuitive implications. The concept of infinity itself and the theorems connected with it are, generally, surprising for an intuitive approach. Infinity exceeds every real human (mental or practical) experience. Only by an intuitive leap can infinity get some meaning. On the other hand, our intuitions are normally shaped by our daily experience, dealing with finite objects and events. As we have said above, the main purpose of our present research was to check the possibility of measuring the degree of intuitive acceptance of a certain solution or interpretation. Infinity seems to us to be an adequate touch stone for validating the meaningfullness of such an operation.

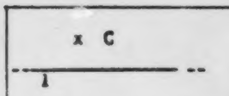
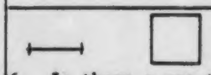
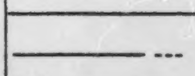
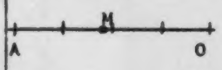
The only problem which was not directly connected with infinity was problem 3 (equality of the opposite angles formed by intersected lines). We have introduced this item because it seemed to us to possess a high degree of "immediacy". If our sets of check questions work well, that item has to range very high on the scales of confidence and obviousness.

Our purpose for including that problem was than to have such a point of reference outside the realm of the other problems and which we could safely consider as being highly intuitive.

For the remainder of this paper the numbers used to indicate problems are those which appear in Table 1 rather than the numbers designated in the presentation of the procedure.

Table 1: Frequencies, Levels of Confidence (C), Obviousness (O) and Intuitive Acceptance of Various Types of Solutions

		I			II		
	answer →	$d_1=d_1$ N=102 %=95.0			$d_1\neq d_2$ N=2 1.8%		
		C	O	I	C	O	I
1. Comparison of the angles L_1 and L_2	N=108	3.28	4.03	3.46	3.5	6.00	4.6
	answer →	"No" N=90 %=84.1			"Yes" N=15 %14.0		
		C	O	I	C	O	I
2. Will the process come to an end ?	N=107	3.18	3.84	3.25	1.60	2.60	1.71
	answer →	"Yes" N=52 %42.5			"No" N=51 %=48.6		
		C	O	I	C	O	I
3. Is there a one-to-one correspondence?	N=105	2.92	3.80	3.07	2.50	3.40	2.62
1,2,3,4,.....	answer →	"No" N=13 %=12.4			"Yes" N=87 %=82.9		
		C	O	I	C	O	I
4. Is there a one-to-one correspondence?	N=105	2.58	3.45	2.98	2.94	4.02	3.22

		I			II					
	answer →	"Yes" N=97 %=90.7			"No" N=9 % 8.4					
		C	O	I	C	O	I			
5. Is it possible to draw a parallel to l through point C?	N=67	2.58	3.58	2.81	2.44	2.55	2.42			
	answer →	"Yes" N=40 %=38.1			"No" N=57 %54.3					
		C	O	I	C	O	I			
6. Is there a one-to-one correspondence?	N=105	2.12	3.07	2.24	2.01	2.80	2.18			
	answer →	"Sum=2" N=6 %5.60			"Sum =∞" N=55 %=51.4			Sum<2 N=18 %=16.8		
		C	O	I	C	O	I	C	O	I
7. The sum of the segments	N=107	1.00	2.60	1.25	2.63	3.60	2.79	2.89	3.41	2.91
	answer →	"It depends" N=7 %=6.5			"Yes" N=83 %77.6			"No" N=15 %=14.0		
		C	O	I	C	O	I	C	O	I
8. Shall one of the points of division hit point C?	N=107	0.50	3.30	0.86	1.88	2.78	1.90	1.89	2.78	1.90

Results and discussion

The results are summarized in Table 1. Column I refers to a category of answers which are considered, generally, at the school level, as "correct". For instance, to the problem concerning the parallel line (Pr 1) there is more than one answer which is mathematically acceptable. But, because the geometry taught in schools is usually , the Euclidean Geometry, the uniqueness of the parallel is considered as being the "correct" answer.

We would like to stress here again that, what is important for us is the answer as such and not its characterization as "correct" or "incorrect". We are aware on the fact that the "correctness" of a solution can be determined only in reference to a certain system of axioms or to a certain model. In fact, our main interest was to determine the degree of "intuitiveness" of the solution for the subject itself. Therefore, when referring to "correct" or "wrong" answers these qualifications mean simply: in reference to what is generally taught at the school level".

In the second part of the table there are the answers which would normally be classified as "wrong" (though, again, some of them may not be considered "wrong" from a different viewpoint or referring to different axioms). Under C there are the mean scores for confidence level; under O the mean scores for obviousness; under I the mean scores for intuitive acceptance (or "Intuitiveness"). The level of intuitive acceptance has been calculated for each subject and each answer by using the formula:

$$I_n = \sqrt{C_n \times O_n}$$

Therefore the I score combines the C and O scores. Under I there are the mean values of the I scores. In Table 1 the problems have been ordered according to the I values of the "correct" answer.

We have supposed that those answers which are easily (intuitively) accepted will present also high frequencies. As it can be seen this was really the case, except problems 4 and 5. The problem which scored the highest value of intuitive acceptance (we refer to the correct solutions) is the intersection problem (equality of opposite angles). This fits our previous supposions concerning the high degree of intuitive acceptance of that sentence. The same problem presents the highest percentage of correct answers. At the other end of the range we find the problem referring to the division of a segment. A very small number of subjects gave the correct answer and, at the same time, that problem has the lowest mean score of intuitive acceptance. This result also fits what could have been predicted, i.e. the solution to that problem refers not only to the infinity of a process but asks some concrete representation of the result of that process in a set containing an infinity of elements. With a few exceptions, levels of confidence, obviousness and intuitiveness follow the same order. The exceptions will be discussed later. It can also be observed that the C mean scores are always lower than the O mean scores. This is probably an artefact caused by the structure of the items. What is worth remarking are those situations in which there is a striking difference between the C value and the O value. It may be observed that the difference increases with the decrease of the "intuitiveness" of the solution. At the end of the range (problem 8, the infinite division of a segment) the situation becomes strikingly contradictory; an extremely low level of confidence in spite of a high level of obviousness. Using the formula $\frac{O-C}{C}$ for expressing the relative difference between the two measures we find the values 0.23; 0,20; 0.30; 0.33; 0.39; 0.45; 1.60; 5.60, for the 8 problems as they are ordered in our table.

At large, the problems, except two, can be classified into three categories. Certainly, the number of problems we used was too small for offering a complete idea of what these categories may be. But, as it will be seen, that tentative acceptance seems to express some basic alternatives in the dynamics of intuitive acceptance.

a. In the first category there are those problems which present a high frequency of correct responses, a high degree of intuitiveness, and the values of C and O relatively very close.

This is the case with problems 1 and 2. Problem 1 refers to the comparison of two opposite angles and problem 2 refers to the operation of successively adding to a first segment AB, segments having the lengths ½, ¼, 1/8, etc. of AB.

The two problems belong, clearly, to two completely different domains.

The first is almost reducible to a perceptual representation while the other belongs to the domain of infinity - for which no perception and no practical experience may be really helpful. In spite of that basic difference both questions and the respective answers present a high degree of intuitive acceptance.

Let us consider now the main types of justifications offered by the subjects for their solutions to problems 1 and 2. With respect to problem 1 the subjects quote generally the corresponding therem: "The opposite angles formed by two intersected lines are equal". Some of the subjects add the geometrical proof.

There is a very small number denying the equality of the angles. It seems, simply, that they did not understand the question. For example:

"It depends on the point of intersection. Sometimes the angles are equal some times they are not."

Let us consider now some typical justifications to problem 2. Most of the subjects, as it has been seen, admit the infinity of the process.

"It is always possible to add a line segment which will have the half of the length of the former line segment"

"A line consists of an infinity of points and each segment could be divided an infinity of times. Therefore we can continue to add segments".

"A line is an infinity of points and it is always possible to add other points".

"It is always possible to add a smaller line segment".

"½, ¼, 1/8, 1/16 ---- The numbers have no end."

"For each number it is possible to find a number which will be the half of it and therefore the process has no end".

It may be seen from these examples that, generally, the subjects believe in the infinity of the process.

There are two main types of justifications. One category refers to the fact that a segment can be divided indefinitely. The second category refers to the fact that a line segment can be extended indefinitely. Both categories accept without questionning the infinity of a process.

It may be claimed that the subjects refer to some learned truth in order to justify their answers. But what is really relevant is that all these subjects have accepted the infinity of the process, i.e. a notion which is beyond every possible control, with nearly the same degree of confidence and obviousness as they did when solving problem 1, which is completely under perceptual control! This finding seems to us to represent an interesting experimental conformation of Poincaré's interpretation. Referring to the recurrent reasoning he has written:

> "Why this kind of reasoning imposes itself to us with such an irresistible evidence? It is because it expresses the capacity of the spirit, which knows that it is able to conceive the indefinite repetition of the same action after learning that this action has been once possible. The spirit has a direct intuition of that capacity and the role of experience is only to provide the opportunity for becoming aware of it" (Poincaré, 1906, pp. 23-24).

It is really fascinating to find that we are ready to confer to suppositions which, certainly, exceeds every possible verification, such degrees of credibility and self-evidence which are so close to those assigned to a perceptual representation.

b. The second category refers to problems which present a split in the main types of answers. The frequencies of the two categories of opposite answers are relatively close.

In our questionnaire there have been two problems in that situation. In fact, they are similar. Question 3 asks on the possibility to establish a one-to-one correspondence between the points of a line segment and the points of an (infinite) line. Half of the subjects answered "Yes" and half answered "No". Both categories are confident in their answers. The difference between the levels of confidence, obviousness and intuitiveness are relatively small. Let us consider some of the explanations offered by our subjects. First, the question referring to the correspondence between the set of points of a line segment and the set of points of a line.

A) The elements of the two sets may be put into one-to-one correspondence, because:

> "There is an infinity of points in both."
>
> "Practically, it is not possible because the line segment is limited. But there is an infinity of points in the segment and for this reason there will be always points to fit the points of the line."

Hence, the basic idea is that the one-to-one correspondence is possible because there is an infinity of points in each of the two sets. Let us quote another type of justification:

> "The point has no magnitude and therefore it is possible to establish a correspondence between a big point of the line and a small point of the line segment."

We have here a strange and interesting mixture of a conceptual and a concrete interpretation of the notion of a point. For these pupils to have no magnitude means "to have no determined magnitude."

Let us see now how other subjects justify the answer: "There is no one-to-one correspondence between the points of a line segment and the points of a line";

"The number of points of the line is bigger than the number of points of the line segment. The line is continuing to infinity and the segment is limited."

"A line segment, no matter its length, is a part of a line (this is the definition of the line-segment) and for that reason there are less points on the line segment."

In short, two types of justifications appear. There is no possible correspondence because: a) one set is larger than the other; b)"because one set is only a part of the other." The fact that both sets contain an infinity of elements did not affect the subjects decisions.

In our opinion, these findings reflect the internal contradiction of the intuitive attitude itself when facing a one-to-one correspondence problem with infinite sets. It is intuitively natural to affirm that the points of a line segment and those of an (infinite) line can be put into one-to-one correspondence because both are infinite sets. It is also natural to affirm that the sets of points of a line segment and those of an (infinite) line cannot be put into one-to-one correspondence because the segment is limited and the line is infinite. The ordinary pupil has not, generally, the feeling of a cognitive conflict when reacting to that question. The feeling of conflict, of internal difficulty, becomes accute for people who try to grasp the entire complexity of the problem.

What concerns the comparison between the set of points of a line segment and the set of points of a square - the frequency of "No" answers is higher than the frequency of "Yes" answers but both are not far from 50%. It seems to be more difficult to accept the one-to-one correspondence between a one dimensional and a bi-dimensional figure. Almost the same categories of justifications can be found here as in problem 3. A one-to-one correspondence can be established between the points of a line segment and the points of a square because ;

"There is an infinity of points in the line segment and in the square."

"On the line segment there is an infinity of points and therefore they can be put into correspondence with the points of the square."

"Theoretically, the correspondence is possible, but practically not."

"On the square there are big points and on the line segment there are small points, therefore they can be put into a one-to-one correspondence."

As it can be seen, here also, the answers belong to two main, opposite, categories: those who affirm that a one-to-one correspondence is possible because in both sets there is an infinity of elements and those who affirm that the correspondence is not possible because the square is bi-dimensional and the line-segment is uni-dimensional.

There are more subjects who reject the possibility of a one-to-one correspondence (N=57; 57%) than those who accept it (N=40; 38.1%). Both categories present a relatively low level of intuitive acceptance.

According to that criterion the "Yes" answers occupy place 6 and the "No" answers occupy place 7, in the series of 8 problems. On the other hand, it can be observed that the levels of intuitive acceptance, for these two opposite categories, differ only slightly (2.24 for "Yes" and 2.18 for "No").

Here again, we have a situation in which intuition is tempted by two contradictory attitudes. Generally, the subjects choose only one of them. But there are also subjects who try to overcome the difficulty by considering that the points may have various magnitudes. The rather slight difference between the degrees of intuitive acceptance of the "Yes" and the "No" answers is characteristic for such contradictory situations.

c. A third category of problems includes those which receive a very small number of correct solutions and which present a very low level of intuitive acceptance for these correct solutions. On the other hand, for the same problems, there are high frequencies of wrong answers, the acceptance of which is higher than that of the right answers: This is the case with problems 7 and 8.

Let us consider first the problem 7. Only 6 subjects (5.60%) answer correctly : "the sum of the segments is equal to 2." Three of the subjects do not explain their answer and three refer to the sum $1 + \frac{1}{2} + \frac{1}{4}$ ----

It is then, interesting to notice that though the answer of these subjects is mathematically correct they do not feel themselves satisfied with it. The most frequent type of wrong answers is constituted by those who affirm that the sum of the segments is infinite. Let us quote some justifications:

"A line can be extended endlessly."

"We shall always be able to add another part."

"There will be an infinity of segments."

"The process can be continued endlessly."

"On a segment there is an infinity of points."

The main idea is that the process is infinite: there will be an infinity of segments and consequently the sum of the segments is infinite. It is a clear, logical conclusion, perfectly fitting our primary intuition. If we refer to the column of noncorrect answers, we can see that the answer: "The sum of the segments is equal to infinite", occupies the third place on the scale of intuitive acceptance.

We have here an example of what can be termed: a counter-intuitive situation. The correct answer is offered by very few subjects and even those are far from feeling safe with it. On the other hand, the majority of the subjects give a typical, erroneous answer and, moreover, that answer is considered as possessing a relatively high level of intuitive acceptability.

There is also a third category of answers to problem 7: there are subjects who affirm that the sum of the segments will only approach 2.

"S = 1,999 ---. The sum tends to 2 m."

"The sum tends to 2 m. No matter how much we shall continue to add segments, we shall never reach 2 m."

"Summing up, halves and halves of the halves etc. we shall never reach (the lengths of) AB."

"$S = 2 - \frac{1}{\infty}$ because there is no end to the sum of the segments."

There are 18 subjects who gave that answer. The level of its intuitive acceptance is relatively high (2.91). Putting together those who answered: "S=2" and those who answered: "S ≠ 2 but tends to 2", we get 25 subjects i.e. about a quarter of all the subjects. The affirmations that "the sum equals 2" or "tends to 2" have probably been learned by most of the subjects during their mathematical courses.

But, while for those who affirm, that the "sum equals 2" their answer presents a very low level of intuitive acceptance, for those who claim that the "Sum only approaches 2" the level of intuitive acceptance is 2.91. (close to the top of the range). Such a situation seems to us to confirm the utility for seeking a measure of intuition acceptance. It is not sufficient to learn that a subject remembers a certain answer or even that he accepts it as being the correct mathematical answer, in order to know, how deeply, how fully, the intuitively agrees with it.

Why is the answer "The sum approaches 2" offered by more subjects and especially why . is that answer accepted with much higher intuitive understanding? The explanation is simple enough. For our usual logic and for an intuitive understanding, infinity is only a potentiality. An actual infinite, (as it appears in the Cantorian approach) has no intuitive representation. The concept of actual infinite is intuitively contradictory.

At a first glance, problems 4 and 5 can not be included in any of these categories. Problem 4 asked if it is possible to establish a one-to-one correspondence between the set of natural numbers and the set of points of a line. Only 13 subjects (12.4%) give the correct answer, though the degree of acceptance of the answer is in the middle of the range. On the other hand 82.9% give a wrong answer ("It is possible to establish a one-to-one correspondence etc.") and that wrong answer presents a high degree of intuitive acceptance (I=3.22). It is interesting to observe that the degree of acceptance of the answer "It is possible to establish a one-to-one correspondence between the two sets" is almost the same as the degree of acceptance of the answer "The process of adding segments is infinite."

Let us quote some explanations. The typical explanation offered by those who claim that the one-to-one correspondence is not possible, is the following: "There is a infinity of points on the line, and there is an infinity of natural numbers." There are other justifications which mix mathematical concepts with concrete interpretations. For instance:

> "The line can be divided in segments and to each segment we can assign a number."
>
> Or:
>
> "Because the point has no dimensions, we can make points of any size we want."

Let us now consider the correct answer: "It cannot be established a one-to-one correspondence between the set of points of a line and the set of real numbers." There are some few subjects that offered a correct explanation (though, of course, incomplete).

> "The set of points on a line includes the set of natural numbers but there are also other numbers."

Most of the explanations prove that the, apparently, correct answer is in fact, based on wrong interpretations.

> "The line has no beginning but the numbers start with 1."
>
> "A line segment consists of many points which are not visible and therefore there are not enough natural numbers."

As a matter of fact, most of the explanations offered by our subjects for their, apparently, correct answer prove, that, their solutions are based on completely wrong intuitive interpretations.

Therefore, these answers have to be considered as wrong, atypical ones. The really relevant finding is that for a great majority of subjects the set of points of a line and the set of natural numbers can be put into one-to-one correspondence. Moreover, these subjects accept that solution as highly evident and reliable. The solution: "The points of a line and the set of natural numbers cannot be put into one-to-one correspondence" appears as an anti-intuitive one. Therefore problem 4 has similar characteristics as problems 3 and 8 - the frequency of erroneous answers strongly exceeds that of the correct answers and the degree of intuitive acceptance of the erroneous answers exceeds that of the correct solution. What is specific to the problem is the fact that the degree of intuitiveness of the wrong answers is very high (I=3.22).

Question 5 refers to the possibility to draw a parallel to a given line. As it could be expected, a very high percentage of subjects have answered affirmatively. All of them have learned the famous axiom . What it is surprising is that the degree of intuitive acceptance is not very high. It occupies place 5 much beneath the degree of intuitiveness of problem 2 (the equality of opposite angles) though both are well know to the pupils from their mathematical courses. Let us analyze the subjects' arguments. Some of them simply affirm the possibility of drawing the parallel line (as invoking an evident truth or a learned truth).

For instance: "It is possible to draw a parallel through point C. Two parallel lines do never intersect." "It is possible to draw a parallel through C to line a, which will not intersect a, in conformity with the definition of parallel lines."

Some of the subjects explain the possibility to draw a parallel line referring to the graphical representation:

> "It is possible to draw (to sketch) a line b which will not intersect a because C does not belong to a."

Those who claim that the two lines must intersect, generally, present an aberrant explanation. For instance: "We do not must draw a line through line a and it is impossible to draw a line only through a or through C."

As we have already said, the intersting point is that a very known, learned statement which seems, at a first glance, to correspond so directly to our intuitive demands, presents finally, only a middle degree of intuitive acceptance (we refer of course, again to the correct answers).

It is not because Euclid 's axiom deals with infinity that the statement is considered less intuitive than,for instance,the therem referring to the equality of opposite angles. The statement expressing the possibility to draw, endlessly, line segments which are more and more longer, presents a higher degree of intuitive acceptance. The incertitude is probably caused by the fact that the subjects feel that the statement includes some thing more than the acceptance of a potentially infinite operation. The statement refers to a specific event: the possible intersection, somewhere, of the two lines. This is, in fact, something more than the pure intuition of the possibility of an infinite continuation of a process.

Question 5 cannot be included in any of our above three categories. This question presents a very high frequency of correct answers but only a middle degree (2.81) of intuitive acceptance, expressing middle degrees of confidence and obviousness. (As we said, the "No" answers do not represent some options for a non-Euclidean geometry but, rather, misunderstandings).

As a matter of fact, that finding is a psychological-experimental confirmation of a truth which has been discovered by the history of mathematics only of hundreds years of efforts: The famous postulate of Euclid is not an absolute truth. It does not have any empirical support and does not express an absolute necessity of our mind. We tend to accept it intuitively at a first glance, but it can be replaced by statements contradicting it without violating the formal laws of thinking.

Summary and final comments

In order to improve the effects of mathematical education care must be taken to identify the real sources of difficulties the pupils encounter in learning mathematics.

A main source of such difficulties has to be sought in the basic intuitive patterns he uses when solving mathematical questions.

Despite the fact that mathematics has a pure logical structure we cannot avoid the intervention of intuitive interpretations when coping with mathematical situations. Intuitive representations and interpretations are active when we are striving to understand, to solve, to memorize or to create in the field of mathematics. Therefore, we must suppose that many difficulties in learning mathematics are generated by a lack of adequate intuitive representations or by the distorting effect of poor, non correct or unadequate intuitive interpretations.

The main purpose of the present research was to check the possibility of measuring - as objectivety as possible - the degree of immediacy of a certain intuition i.e. the degree of the subjective, direct, intuitive acceptance of a certain solution by the learner himself. Such an enterprise, if validated,may have important practical consequences. Let us consider for instance that some typical error is caused by an intuitive pattern strongly accepted by the learner. In such a situation,more verbal explanation and more standard practices will not help. The intuitive background has to be re-elaborated. On the basis of some preliminary investigations we concluded that the immediacy effect - i.e. the feeling of intuitive acceptance - is produced by the convergence of two ingredients; namely, the level of confidence and the feeling of obviousness. Though, generally, correlated, these two components cannot be reduced one to the other. In extreme cases they can even be separated. In some interesting situations, conflictual relationships between conviction and obviousness can be described.

For measuring the degrees of confidence and obviousness we used two groups of check questions, one group for each of these two aspects. A hierachical analysis proved that each of the two groups of questions has a Guttman scale structure. The scores were weighted according to their rank in the scale.

The square root of the product of the two obtained scores was defined as expressing the degree of immediacy - or intuitive acceptance - if the solution by the subject himself.

According to the frequencies of the main types of solutions and to the degrees of their intuitive acceptance by the subjects, three categories of problems have been described.

1. A first category includes the problems which:

a) get high frequencies of correct solutions (more than 80%) and b) these solutions present for their authors, high degrees of intuitive acceptance.

We may characterize that category of problems (and the corresponding solutions) as being highly intuitive. It may be supposed that problems belonging to that category do not cause difficulties to the teaching process. (Problems 1 and 2).

2. Problems which get two main types of opposite solutions, the frequencies of which are close to 50%. The degrees of intuitive acceptance of these solutions are situated in the middle zone of the range. (Problems 3 and 6) These problems and the possible solutions to them, can be described as internally contradictory. (from a psychological viewpoint). Such problems may, probably cause didactical difficulties. But we may expect that a good didactical strategy can increase the frequencies and the levels of intuitive acceptance of those solutions which are commonly considered as correct ones.

3. Problems which get low frequencies of correct solutions and high frequencies of typical non-correct solutions. At the same time, the non-correct solutions present higher degrees of intuitive acceptance than the correct ones (Problems 4, 7 and 8).

The problems belonging to that category and their correct solutions - may be qualified as counter-intuitive. Intuition does not generally accept them. Instead, the opposite answer, the answer considered as non correct in reference to the usual school mathematics, - appears commonly to the pupils as being acceptable.

We may expect that the regular teaching process will have only a temporary superficial effect, or even any effect at all, on the basic intuitive approach to that category of problems. The intuitive background of the typical, non-correct answers is so strong, so deeply rooted in the pupils mental habits and ways of interpretation that it is difficult to produce a lasting modification.

A previous research concerned with the notion of infinity (Fischbein et al., 1979) included some of the items which were used again in the present research. One of these items was problem 8 (will one of the points of division of a segment AB hit point C? etc). The percentages of correct answers were the following: II - 6.5; III - 6.9; VII - 12.5; VIII - 12.5; IX - 8.3 (the roman figures represent school grades). In order to answer correctly, the pupils have to know that the points of a line segment correspond to rational and irrational numbers and, therefore, the correct answer is: "It depends on the location of the point". Though, at least starting from grade 7, the pupils have learned about rational and irrational numbers, this fact did not influence their basic interpretation of infinity: by continually dividing a segment, every point can be reached. For our natural intuition there are no levels of infinity. The Cantorian approach is essentially counter-intuitive. We accept easily the idea of infinity (see problem 2) but we are not ready to accept intuitively that one infinite can be richer than another. Intuitively, it sounds absurd. Even those subjects who have answered correctly, i.e. that the solution depends on the location of C, even those, do not believe strongly in their answers.

As it can be deduced from the data referring to our previous research (Fischbein et al 1975) there is some progress from grade 6 to grade 7 (6.9% versus 12.5% of correct answers) But the progress stops here.

In Grade 9 there have been even less pupils who answered correctly than in grade 8. Consequently we submit that such counter-intuitive situations are only slightly influenced by the usual teaching process. A correct interpretation taught in school, which contradicts a basic intuitive approach, has little chance to be remembered and to be efficiently used, especially in non-standard situations.

The same intuitive pattern concerning infinity determines also the high frequency and the high intuitive acceptance of the non correct answer to problem 4: most of the subjects strongly consider that the set of natural numbers and the set of points of a line can be put into one-to-one correspondence. The basic intuition is that infinity means only non exhaustible. Therefore, infinite sets must always be equivalent.

An interesting situation occurs when two strong opposite intuitive interpretations could be elicited by the same circumstances. It was the case with problems 3 and 6. The set of points of a line segment and set of points of a line (prob. 3) or the set of points of line segment and set of points of a square (probl. 6) can fit because in both questions there are pairs of infinite sets. On the other hand it seems intuitively natural to affirm that, in both comparisons, one set is larger or richer than the other (oneincludes the other). Our subjects are not, generally aware of the conflict. They simply react in conformity with one of these two basic intuitive approaches.

Superficially we could have closed here the discussion on this last point. But inspecting the data we find that the degrees of intuitive acceptance are distributed in the middle zone of the range. The effect is stronger at problem 6 where an one-dimensional and a two-dimensional figure are compared. Thirty eight percent of the subjects affirm that the two sets can be put into one-to-one correspondence - thus following their intuitive bias: all infinities are equivalent. As we said, the subjects are not aware of the intuitive conflict. Therefore we could have expected a full intuitive acceptance of that solution. But, as a matter of fact, the intuitive acceptance of that solution by the subjects is rather low. This fact is, in our opinion, an indicator of the latent intuitive conflict mentioned above. In other terms: when analyzing the intuitive infra-structure of a body of knowledge we have much to learn not only by detecting the corresponding intuitive biases but also by getting some measure of their degrees of acceptance by the subjects themselves. The frequency of a certain category of answers - though representing an useful information for detecting an intuitive bias - does not express all the information needed for qualifying that bias. A low level of intuitive acceptance of a certain solution, for instance, may constitute a reliable witness of an existing intuitive conflict on which the subject himself may not be aware. Didactical strategies must be devised for shaping improved intuitive interpretations and for overcoming conflictual intuitive biases.

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HOW DOES REFLECTIVE THINKING DEVELOP ?

HANS FREUDENTHAL

0 introduction

About a year ago I suddenly realised that in the past when observing children I might have overlooked precious evidence. Did I look consciously enough for symptoms of reflective thinking and of growth of reflective thinking? - this was my problem. The idea of a possible shortcoming occured to me when I was preparing a paper for a conference the theme of which was 'Proof in Mathematical Instruction'.

I had set out to consider the activity of proving not as an isolated activity with roots of its own but as the acuminated stage of activitities that develop naturally and might be developed didactically. The prestage of proving I imagined was constructing, which if it happens mentally, means implicitly proving the truth of assertions of existence. But if constructing was a prestage to proving, there must be an intermedial stage, which I decided was reflecting.

Deductivity as a principle started in Greek geometry. Geometry,however, is older than the Greek. About two millenia before Euclid the so-called Pythagorean theorem was known to Babylonian scribes. Did they discover the theorem empirically ? Of course not. Somehow they proved it - there are many ways to do it.

So what did the Greek invent ? One of the first theorems in Euclid's Elements is the equality of the angles at the basis of the isosceles triangle - not a *hidden* truth as the Pythagorean theorem but an *obvious* one - obvious if you may trust your eyes. Why did the Greek prove theorems like this ? Did they not trust their eyes ?

Symmetry is the most obvious and the most striking geometric feature. You can trace it back to Stone Age decorations. Symmetry tells us about the equality of the angles in the isosceles triangle, about the diameter halving the circle, about the diagonals of a parallellogram bisecting each other. All theorems reliable sources ascribe to the sage Thales are about symmetry. Tradition says he proved them. What did this mean at the dawn of geometry? I think at least that he reflected about whether what his eyes told him was true.

Geometry is the domain where deductivity took its start. There is so much obvious in geometry that the questions 'how do I know what I know ?' is in no field of cognition less obvious.

How do you know that the diagonals of a parallellogram bisect each other? Have you ever reflected on it? You need not. Indeed, you can prove it by congruent triangles and suchlike devices. But you knew it long before you learned about congruent triangles. I am sure you did, because I observed little children showing they know it. Well, they cannot tell you why, as little as you can tell me how, long ago, you knew such facts.

This is the way, I think, deductivity started in history. It was not that people did not trust their eyes but that they asked themselves 'how do I know what I know ?', or for that matter, one asked the other 'how do you know what you say you know ?' Meanwhile Greek geometry has been created and codified. We need not any more ask ourselves why we know what we know. Teachers and textbooks tell us the answer before the question arises. This is the way geometry is being taught, or rather, was being taught, and moreover the reason why it is not taught any more.

In the lecture I alluded to I ventured to affirm that nevertheless even now people are asking themselves and others how they do know what they say they know, and availing oneself of this natural human attitude would be paving the way to the activity of proving in mathematical instruction. From this point of view I considered reflecting as a prestage, the indispensable prestage of proving. Let us learn from history, I said: proving not as an independent track, but as formalised reflecting.

1 reflection unfolding

Reflecting is often used as a mere synonymous to thinking, but obviously if I speak of reflective thinking, I mean something more sophisticated than mere thinking. Before defining it, I will illustrate it by examples, most of which from the diary of a boy, now nine years old, whom I have been observing from the age of two years onwards - Bastiaan, born 27 April 1970. When preparing the present lecture, I ransacked this diary more systematically than ever before.

14 January 1973. We took our walk. Bastiaan found the top of a bike-bell. When he dropped it, it gave a sound. So he dropped it repeatedly. He threw it, he kicked it. I found it dangerous. 'Let us to go to New Clarenburg' (a building with a broad terrace), I said and put the bell into my pocket. At New Clarenburg he asked for the bell. I warned him that it could fall down from the (one meter high) terrace, which eventually happened. Bastiaan looked after the bell and then said 'So fetch, down the stairs' while showing the way with his hand. This took place at a distance of about 10 meters from the stairs. Not only did he make a plan how to recover the bell, he also disclosed it verbally.

Is this reflecting? I think not. It is anticipating, and the only remarkable feature about it is the verbalising of the mental action.

23 March 1973. I noted down some idiom such as 'what do you say, Grandpa ?' if he does not understand me - actually in adult language it has a meaning that in English should better be rendered by 'I beg your pardon ?' Or: 'For this Bastiaan must grow bigger'(for instance if a wall is too high to climb on). Certainly these are no examples of reflection. But the following is a different case: On a Friday he asked me: 'Does Grandpa go to the Institute tomorrow ?', and only when I answered 'No, it is Saturday tomorrow', did he continue questioning me whether I would take a walk with him the next day. If this way of asking had not struck me as remarkable in his development, I would not have noted it down. Is it reflective thinking ? Not yet, I think, but it is some important prestage.

I cannot but inserting here an episode I have told many audiences many times albeit in other contexts.

26 September 1973. A week ago behind the Military Hospital we were passed by a wheelchair with a woman in it, pushed by a nurse. Bastiaan asked: 'What did the lady say to the lady that pushed the wheelchair ?' After the second 'lady' he hesitated before he, hurriedly, ejaculated the relative clause.

The hesitation indicated that he felt the oddity of the double use of 'lady' and the necessity of the explaining relative clause.

Was the hesitation a symptom of reflecting ? I would say it was - reflecting on one's own speech rather than on one's own thought.

2 December 1973. At that time (and even much longer) Bastiaan's hobby was shopping, in particular buying little cars. He used to buy presents for other people in order to play himself with them. On my birthday (17 September) I must go with him to a shopping centre to buy a present for myself. He insisted on my choosing a little car but finally I had him make the choice. Only a few times did he play with it, but repeatedly he tried to make sure whether I played with the car, still standing on my desk. Now, two and a half months later he does not touch the car anymore nor does he give anymore presents to be used by himself.

Is this a symptom of reflective thinking: doubting whether somebody has the same likings as oneself ? I am not sure. Anyhow it is some prestage.

26 December 1973. It happened a few days ago that we saw a large truck, with a registration number BB ... (cars and trucks are his hobby). I drew his attention to the registration sign. Somewhere at the truck he discovered a 'B3'. I said: 'It looks like a car of Bastiaan's'. Bastiaan: 'Why do you think the car is mine ?' My answer: 'Why do you think I think it is yours ?' Bastiaan: 'Because there are so many B's on it.'

Is this reflective thinking ? I think, yes, though stimulated rather than spontaneous.

Two wheelchairs at a canal. 'What is this top for ?', Bastiaan asked, showing to a screwcap at the back of one chair. I did not know. 'Is it driving on gasoline ?', he asked. I comfirmed it. 'Then it is for the gasoline', he said.

A striking episode. Though he already guessed that the screwcap covered a gasoline tank, he did not disclose his hypothesis until he had gathered more evidence to support it. It is a mental reservation: as long as you are not sure about a conclusion, look for more evidence. How consciously did he make this reservation ?

30 December 1973. The day before yesterday when passing a garden midget, he asked 'what does the midget say ?' I answered: 'Bastiaan, what a little man are you'. He continued. He is a little man himself'. Yesterday at the same spot, Bastiaan asked: 'What does the midget say to you ?' I answered: 'What do you think he says ?' Bastiaan: 'What a big mister are you.' And after a hesitation he continued: 'Grandpa, why does he say 'what a big mister are you ?''It is a pity that I did not immediately turn to the counter question: 'Why do you think he says so ?' He would very likely have answered: 'Because you are so big.'

3 January 1974. On the central post distribution yard we often watch the operations of the lifting trucks carrying high trolleys containing post-bags. Yesterday when we arrived, there was an open truck with still a few trolleys left in the background. I first said there were three, then

made it two and explained: 'I first thought there was another behind that one - it looked so.' Bastiaan : 'There are three, there is one behind.' I insisted on two. When the truck was emptied, there were three. Bastiaan: 'I knew it.'

7 January 1974. Walking 'between the rails', stepping from one tie to the next. When he missed one, he explained: 'It is too large.' We are approaching a switch. He is walking with his feet in the space between the fixed and the movable piece. He had done so for many times in the past. For the first time he commented on it: 'It is getting ever narrower, and then I will step out.' It is surprising again and again how Bastiaan is commenting all he is doing. One can see it on his face before he opens his mouth that he is thinking about something.

I took this over, not as a convincing example of reflective thinking, but in order to witness the degree of consciousness in his thinking.

30 January 1974. I told him about a big toy house made for the eldest brother of his mother by his grandfather - that is my father-in-law - for his first birthday. Afterwards the other children also played with that toy house, and finally his mother who was the youngest. Bastiaan asks : 'Why ?' I answered: 'What do you think ?' He: 'Because they were little brothers.' He wanted to know where the toy house is now. I told him I thought Grandma gave it to a school when nobody was left to play with. 'Indeed, does Mirjam (his mother) still play ?', I said, 'No', he answered, 'she cooks'. 'And Frans (his father) ?' 'No, he works.' 'And Grandpa ?' 'No, he does mathematics.' (He once saw me draw geometrical figures on the margin of a newspaper and asked me what it was. I said, mathematics. Triumphantly he ran to his grandmother to tell her that he will do mathematics.) 'And Grandma ?' 'No, she works.' 'Why didn't you keep the toy house for me ?', he continued. 'We didn't know that there would be a Bastiaan.' Again and again he inquires about the toy house. 'Perhaps the school does not exist any more.'

Compared with the episode of 2 December 1973 the differentiation of activities between adults and children has become clear.

Because of its close connection to, rather than as an example of, reflective thinking, I mention the first instances of subjunctive conditionals I found in the diary (probably not the first at all in the boy's speech).

5 February 1974 : During our walk he noticed a glass marble and kicked it. 'If I push hard, it would roll upon the road' (which he does not want). A striking use of the conditional though not in the subordinate clause - repeatedly used but mostly either in the principal or in the subordinate clause rather than in both of them. Also in answering questions: 'Else it should ...' A few days ago: 'If it were ...' Another case: Two children in carriages. He said: 'They are little brothers.' (In Dutch this can also simply mean little boys.) I answered: 'No, little brother and little sister.' He: 'If Monica were a boy.' Probably this must be completed in the way: 'We are also little brother and little sister, but if Monica were a boy, we would be little brothers.' Or did he mean: 'As though Monica were a boy ?'

1 April 1974. Bastiaan made an April fool of his mother. A week ago in my car. When suddenly I slowed down, he asked me: 'Why did you do it ?' 'I did not know what the car in front would do, whether it would turn off or park.' 'My father always knows what the cars are doing.' A bit later a similar situation with cyclists. 'Why did you do so ?' I explained why. 'My father always knows what bikers are doing.' On the way back: 'Why do you drive a Peugeot rather than a 'Duck' ?' I told him a story about Peugeots being more comfortable, driving more smoothly. 'My father thinks a Duck is driving more smoothly.'

8 April 1974. The day before yesterday he told me: 'Yesternight when Dad and Mom slept, I got a funny cough.' I asked him: 'How did you know that Dad and Mom slept ?' 'It was already light', he answered. 'So it happened this morning ?', I said. 'Yes', he confirmed. I asked him: 'Are you always awake before your Dad and Mom ?' 'Yes.' 'What do you do then ?' No answer. I continued: 'Do you always cough then ?' He understood it was a joke and laughed.

On a football turf a training short on a pole. Bastiaan noticed it and found it a funny look. Certainly some child forgot it. 'What might his mother have said ?', I asked him. 'Where is your short ?' 'And what did the child say ?' 'I left it on the football ground.' 'And what did his mother say ?' 'Take it back.' 'And the child ?' 'Didn't do it.' And so on.

9 July 1974. Three days ago, back from the museums, we passed by car a spot where some accident had happened: a damaged car and across the road something I could not identify, a tube or a pipe. Bastiaan said about the following: 'It was a streetlamp, and this I could see because I am sitting this side and you couldn't because you are sitting that side. I noted in the diary: Quite characteristic, this reasoning about what others do or can do. Against Piaget's 'egocentrism'.

12 July 1974. As usual he has collected a lot of garbage during our walk. (Actually it does not mean garbage to him: all he takes, he does with an intention.) He asked: 'Did your grandpa allow you to collect such a lot of garbage on street ?'

He wants us to go to the fair. I do not know about any fair. 'My father said there is a fair, and then there is a fair. Some fathers and mothers do not know there is a fair, but my father always knows that there is a fair.'

13 August 1974. (A story I told many times, though in the context of the number concept.) Dinner at Bastiaan's home: at the rectangular table he is sitting opposite his sister, his father opposite his mother, his grandmother opposite his grandfather. Suddenly at the dessert of red currants he lifts his spoon in the greatest agitation and ejaculates: 'So many we are !' Indeed there were six currants on the spoon. I asked him 'why ?', and he answered: 'I see it so', and then 'two children, two adults, two Grandpa and Grandma.' Possibly the six currants lay on the spoon in the same configuration as we occupied around the table, but this I could not see.

At this moment Bastiaan did not yet count. He started counting rather late, that is, a few weeks after this red currant story. The striking feature is that after an obligatory 'I see it so' he could reconstruct the structural isomorphism he had noticed.

Is it reflecting ? I am inclined to affirm it.

I always addressed Bastiaan in adult language. If he did not understand - and this was quite normal - he asked: 'What do you say, Grandpa ?' (See 23 March 1973.) This used to be his formula up to 15 February 1975. On 16 February 1975 I noted: Yesterday he asked me for the meaning of 'elders' (elsewhere); up to now he used to say 'what do you say, Grandpa ?' if he did not understand.

In fact, from this day onwards he always asked for the meaning of a word if there was something he did not understand; he never switched back to his old formula.

It is by reflection that he hit upon the idea that asking for the meaning of a word is a way to decoding messages ? The abrupt change of policy suggests intentionality.

2 speaking with oneself

During the year following the episode of 13 August 1974 and even longer I can find only few symptoms of reflective thinking in Bastiaan's diary. This does not mean there were not more but rather that I did not pay enough attention to them. My attention was entirely absorbed by the boy's development as to number, space, and physical nature.

I will dig up a few episodes from this periode later on. However, I cannot but citing a remark from the report on 'Bastiaan's experiments on Archimedes' principle.[1])

In one of the first lessons we had agreed that before each particular experiment we would try to predict the result - that we understood is what scientists have the habit to do, and he was proud to be one of them - and would try to explain each extraordinary phenomenon. In the report of the fourth lesson (that is the day after the third lesson) I noted in the diary:

24 October 1975. Yesterday Bastiaan told at home he now knew how one could speak with oneself so that nobody could hear it. Certainly this is related to thinking with a view on anticipating the result of an experiment. If he has to predict an outcome, you can see him behave as if he speaks with himself.

Though he described his thinking as mental speech, it is not thinking about words and verbal structure, but thinking about thinking.

3 what is reflective thinking ?

After this row of examples time has come to pronounce myself about what I believe reflective thinking is. For a long time I have been acquainted with a special aspect of this phenomenon in spontaneous and stimulated learning processes. I had developed ideas on a level structure of mathematical learning similar to, and influenced by, but not identical with the Van Hiele levels. In mathematical learning, I claimed, there are levels that in a way

[1]) 'Educational Studies in Mathematics' 8 (1977), 3-16.

but not definitely are determined by the structure of the learning matter: The mathematics performed, acted out on a lower level,is made conscious and analysed and consequently transformed into subject matter on a higher level.

I did not use the term 'reflection' for this higher level activity with respect to the lower one. I am not sure what is the reason why I did not; maybe I spurned a seemingly colourless term where I could provide more substantial descriptions. When wrestling with the problem how proving arises, it was the first time I felt the need for a more compact term, and when I hit on 'reflection', I grasped it was not as colourless as I might have seen it before. If now I use the term, there is an undertone of physics in it: reflecting as mirrors do, mirroring.1)

Indeed, if I use the word 'reflection', I mean mirroring oneself in the other in order to look through his skin, to explore him, to take him in. And consequently since the other is one's like, a human, this is an experience about human behaviour and finally knowledge about one's own behaviour. So from the mirroring oneself at the other it follows as on the night the day, mirroring oneself at oneself, the introspection. It becomes reflecting on oneself, on what one did, felt, imagined, thought, on what one is doing, feeling, imagining, thinking.

Reflecting, once started, is an activity we perform every moment, in order to determine our course of action, or if it becomes an aim in itself, as a mental exercise.

At the age of three, children start asking 'why ?'. It can be meant causally or finally even though the adult may interpret it in a logical sense. The child's 'why ?' may ask for arguments. If, however, the child himself is asked to argue something, it is better asked 'how do you know ?' Perhaps authorities are being adduced, parents or friends. Sometimes in the cognitive field, the answer may be 'I just know it', or if it is representation, 'I see it'. May we find fault with the child for its short answer ? As adults do we always know why we know a thing and haven't we exercised reflection and introspection for a long time ?

---

1) I do not know whether philosophers or physicists were the first to pick up this term. Anyhow Latin 'speculum' for mirror did not produce 'speculation' since 'speculatio' means exploring, but both do have the common root 'specus', which means 'eye socket'.

I coupled the questions of what is reflective thinking to that about its origins. In fact we can tell little about them. Lack of means of expression is one reason, and the other - more important - is lack of attention to how we experience what we experience. Or rather than lack of attention shouldn't I say: impossibility to pay attention ?

Nevertheless I ventured to assert that reflecting in one's own mind, is triggered by that in another's mind. I feel this view is fortified by Piaget's work on the origin and development of imitation - Piaget's best though least known work 1)‚ based on observations of his own rather than of others. Imitation starts reflective, or as Piaget says, circular: initially the little child imitates only such sounds and motions of the adult which are more or less faithful reproductions of sounds and motions produced by the child itself.2) Imitation starts reflective, that is not as selfimitation but mirroring one's behaviour in the other's. What about other utterances, intentions, thoughts ? For instance experiencing intention as such and making it conscious to oneself - when does it start, before or after intention as such is being ascribed to others ? Noticing unexpected acts or unusual behaviour of other people may or may not lead to questions like 'why did you do it ?', 'did you rightly do so ?' or 'would other people do the same in this situation ?' Is it a long step from here to asking oneself the same kind of question ? And even then what is asked for, may be a reason rather than an intention.

There is one argument why reflective behaviour should start with mirroring at the other's mind. The argument is language, or more generally, communication. The child learns saying 'I feel', I want, I think' from the other. Well, before using it, the child must interpret it. How does it succeed ? The analogy with what happens in the development of imitation would indicate that the child succeeds to interpret 'I feel, I want, I think' as its own action because it is prepared to, because somehow it 'knows' what is means. But even then the outside stimulus is needed to make it explicit.

Let us shelve these questions. We simply know too little to answer them.

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1) Piaget, J.:*'La formation du symbole chez l'enfant'*, 1945.

2) The importance of this discovery is unfortunately obscured by such arbitrary and in its application overextended terminology as 'assimilation and accomodation'.

Learning how to observe children is a prerequisite to more knowledge.

4 shifting one's standpoint

Let us turn to more easily accessible questions. Reflection unfolding appears under many aspects. One of them is what I would call shifting one's standpoint - mentally shifting though the standpoint itself may be a local one or a mental one, while the shifting may take place in space, time , or any other, say mental, dimension.

With respect to the 'points' considered let us distinguish a few possibilities of shifting - without any pretence to being exhaustive. One is

shifting from A to B in order to look back to A -

I call it *reciprocal shifting*.

Another is

shifting from A to B while considering C -

I call it *directed shifting*.

And finally

shifting A's environment to B's -

I call it *parallel shifting*.

The most concrete realisation of reciprocal shifting is using a mirror in order to know how one appears to others. A reciprocal shift in time, concretised by shift to an older or younger person: when I grow older, I can climb higher than now; when I was younger I could not climb as high as I can now. Or the mental reservation as reciprocal shift of standpoint: after a supplementary information, I will judge the present situation better.

A directed shift of standpoint, and then in succession, is describing a path. Well, the easiest way to do it, is showing as Bastiaan did to recover the lost bell. But notice that he complemented showing by explaining 'down the stairs'. Monica (4;4) wants to explain some place by means of the walk leading to that point. The forefinger raised but motionless, she exclaims: 'and then you go so, and then you go so, and then you go so ...'. Her wide eyes betray her intuitive vision: lucidly she sees the path in her imagination but she lacks both mimic and verbal resources to describe it. Bastiaan (6;4) in a similar situation shows with his hand which direction to follow 'after crossing the locks'. Describing or explaining a real or imagined situation may require shifting mentally one's standpoint. Bastiaan (9 July 1974) shifted as it were to my place in order to explain why I could not identify the fallen streetlamp. But in general, one may ask whether shifting one's standpoint is facilitated or may sometimes be blocked by assuming another's standpoint. In order to draw a child's attention to some object, one may show to it or turn the child's head in the wanted direction, or move or lift the child (to overcome some obstruction). The child is well acquainted with such procedures - kind of baby language - which may prevent it from learning what is relevant in particular cases, and from developing more sophisticated means. While the sun had just set, Monica (4;8) asserted she saw Venus near the crescent; probably she was right, whereas my eye's acuity failed. She wanted me to sit on my heels in order to see the planet from her - lower - viewpoint. Daphne (4;0) has as long as I remember resisted to moving her body or her head as a device of showing her an object, as she refuses to be helped in almost all physical activities.

Asking preliminary questions before tackling the main question may be also an example of directed shifting the standpoint. Before interviewing the other about a certain subject, one tries to find out where he is standing, and in order to do so one changes one's own standpoint. In the long run this may be the most striking feature of reflective thinking. Reflecting oneself in another may be facilitated by family relations.

The generation shift is an outstanding example of what I named parallel shifting of standpoint. As in Bastiaan's case it may start with the relation between family levels. At a very young age he reflected his relation to his younger sister Monica in that between his parents (reinforced by that of his nearby living grandparents). So, when grown up, he told, both of them would live in a certain now vacant house nearby and would have a little Bastiaan of their own. From the intra-generation relation the stress was soon shifted to the inter-generation relation: when his mother was a child, Grandpa was her father and took walks with her, and when Grandpa was a child, he had also a grandpa. Quite early Bastiaan structured the past, in particular the development of the technique by means of the generation shift. 'How looked cars when you (Grandpa) were as old as I am now, was there T.V. when Mom was my age ?', and so on. Of course, in general young children are not in Bastiaan's opportunity to perform such a clear cut double generation shift. It is perhaps this condition that explains his early differentiating the past. (As a contrast let me adduce a teaching experiment with third graders the vast majority of whom were unaware of the differentiation of the past until they were taught it.)

The generation shift is imperfect. Bastiaan had to learn that by their activities adults differ from children: they do not play. Or don't they ?

A funny observation much later in Bastiaan's life (8;8):

The Dutch word for playing at cards is 'kaarten', which word does not include explicitly a component equivalent to 'playing'. Bastiaan asks: 'Does *kaarten* exist as a real thing ?' The explanation of this strange looking question: *Kaarten* is playing. Playing is imitating the real thing. So what is the real thing you imitate when playing at cards ?

5 lies, jokes, irony and understatements

Years ago I was puzzled by the behaviour of a 7-8 years old who, reading the story of Puzzy in Boots, was not able to grasp that what the tomcat said were lies. The shift required for this reflective thought was apparently less easy than understanding in the story of Little Red Ridinghood that the wolf in Grandma's bed was lying about being Grandma.

Talleyrand is said to have defined language as the means to conceal one's thoughts. Lies, jokes, irony, and understatements are in a way shades of this function of language. I am not sure which of these attributes I should assign to such expression as 'I am sorry, that ... but ...', 'I am awfully sorry that ... but ...', I beg your pardon, but ...', 'Forgive me ... but ...', and so on, where the 'but' introduces some refusal or interdiction, which forcefully nullifies the polite opening. In Bastiaan's diary I noted down such questions of his as 'what asked the mister you to forgive ?' But on one day together with such question as 'why did the gateman (who refused children access to a factory yard) did regret it so much ...?', I recorded the following happening.

27 July 1975. At the end of a long walk with Bastiaan and Monica, at the BP gas station, where among others you can buy ice-cream and toy cars, Bastiaan complained being thirsty. I asked him whether it was not ice-cream thirst. He confirmed it was. I bought ice-cream for both of them. Meanwhile he had set my mind at ease by uttering: 'Though, Grandpa, toy car thirst - such a thing does not exist ?'

It is difficult to classify my own remark either as a joke or an irony, and even more difficult to decide how it was interpreted. Anyhow it is remarkable how Bastiaan responded to my behaviour with a well-conceived counterpart, undoubtedly due to reflexion.

Another confrontation with irony:

15 August 1975. He notices that in the hotelbar (where we often enjoy the magnificent view over the city) you can get lunch. I object: 'it is a *dure tent*' (expensive booth). 'Why is it a tent ?' I explain him that *tent* means a cheap place and *dure tent* is a kind of joke. 'It is as if you are naughty and your mother calls you *lieverdje*' (darling - but most often ironically used). 'She is not allowed by my father to do so', he answers.

About the same time a joke of his own invention:

2 August 1975. On a former highway bridge over the Amsterdam-Rhine-canal, now reserved for pedestrians and bikers they left the now obsolete traffic board '50 km'. He asks: 'Why is it forbidden to walk faster than 50 km ?' The way he asks the question shows clearly it is meant as a joke.

Every time in the future when we pass the spot, he will repeat the question.

At the same walk we find a 500 lire banknote. 'If I give it to my father, he will think we have been abroad.'

It seems that for reasons as mentioned in section 2 I did not continue paying attention to joking in Bastiaan's development for a long time. However, from September 1977 onwards the diary contains many examples of conspiring to fool others. A case of irony or joking:

5 November 1977. Fishbones on street. 'How does it come here ?', he asks. 'Do not ask me', I answer. 'Whom should I ask ?' 'Ask the Lord Mayor.' He picks up the joke: 'Mister Lord Mayor, who owns the fishbones ? Mister Lord Mayor, who owns this leaf ? Mister Lord Mayor, who owns this dog dirt ?' And then he continues by playing the Lord Mayor becoming angry.

14 July 1978. A boat trip through the fortifications of Naarden. The guide intersperses his account with many quips. Bastiaan answers every joke with one of his - too difficult to translate. It looks like a duel between humorists.

He has now entered the period of classical puzzles.

16 July 1978. After I have posed him mathematical problems, he asks me: 'Johnnies' mother has three boys: Quick, Queck and ...' (you are expected to continue 'Quack').

Classics like the 10 sparrows at the roof - three where shot, how many stayed - are obligatory.

22 July 1978. During a walk we saw posters left from a 'puzzle trip': 'How far does the deer stride into the forest ?' 'When do you have as many

eyes as the year has days ?' Of course I must explain the answers. He asks for paper and pencil to note down the questions. I add the question: 'It is the son of my father, it is the son of my mother, but it is not my brother'. After being reformulated with 'your' instead of 'my' he solves the problem.

2 December 1978. He has discovered the secret about Santa Claus and is now playing Santa Claus tricks to his sisters. He explains me that he had recognised and identified the gingernuts his mother had bought and those he found in his shoe at the chimney (a Dutch habit when 6 December draws nearer).

26 December 1978. A joke of the kind of the liar of Creta - too involved to be told in a few lines. The main thing is that Bastiaan understood and enjoyed the structure.

On the contrary, somewhat later Bastiaan did not understand, even after explanations, a joke about people who while trying to excuse, accuse themselves in an ever increasing degree.

6 analysis of errors

It is a pity that for this aspect of reflective thinking I can find only one instance in Bastiaan's diary.

19 March 1977. Bastiaan: 'I was born in 1971, and Monica in 1972.' Actually he was born in 1970, but since I did not notice the mistake, I answered: 'No, then Monica was born in 1973 because she is two years younger than you.' When it crossed my mind that his birth year was 1970, I corrected it and added that then Monica's was indeed 1972. He noticed: 'First I was wrong, and then you', and then continued: 'But it is difficult to figure out when you were born because you are seventy'. Actually I was 71 at this moment, but the celebration of my 70th birthday still dominated his recollections. The conclusion seems justified that first he had figured out he was born in 1977 - 6 (the present year minus his age) and then recognised that he should rather have subtracted the age he would reach in 1977. So I told him I would be 72 late in that year and he figured out I was born in 1905. Of course this led him to ask about the first world war, wars in general, and history.

7 if people do not understand you ...

It often happened that Bastiaan became angry when I did not understand his questions. Lack of understanding, even in adult communication, is often due to lack of context, supposed as trivial by one party yet unknown to the other. So once I explained Bastiaan if I did not understand a communication of his, it could be that I lacked some information he tacitly supposed I possessed.

15 July 1978. Driving along a shop of the 'Jamin chain'. The name Jamin appears twice in Neon letters on the front, horizontally and vertically, but separated. In the horizontal version the letter M is lacking. I drew Bastiaan's attention to it. He said: 'This can only with an odd (number of letters). I did not understand what he meant. He became angry: 'You know very well what I mean.' I told him, as I had often done in the past, that I never lost my patience if he did not understand me. (He should do likewise and think about what was the reason why I did not understand.) Finally he said: 'If they cross each other.' It is clear what he meant: The number of letters must be odd if the words are to be put together to form a regular cross. I should add that at that period the ideas of even and odd strongly occupied his mind.

8 reflective learning at school

Unfortunately this section, which should be the most important, is almost void. Observations of other children would not fit into the present frame. I never observed Bastiaan at school nor did he grant me more than a few opportunities to observe the growth of his scholastic aptitudes.

He fully understands the meaning of the arithmetical operations, he knows how to perform them, though his performance is not yet flawless. He accepts all he learns without any criticism, and in scholastic matters he never asks the question 'why ?', which he is still asking in all other matters. He is not interested in questions like 'why is 3 x 4; 4 x 3 ?' let alone in reasoning about spelling.

However, I think this is a transitional stage. School is a phenomenon he has still to learn to cope with.

Towards a Hierarchy of Understanding in Mathematics.

K. HART.

The Project "Concepts in Secondary Mathematics and Science" (CSMS) based at Chelsea College, London, has been investigating children's understanding of mathematics for the last five years. To do this they have produced twelve test papers each of which dealt with a different topic in the English secondary school mathematics curriculum. Prior to testing a large sample of children, about thirty pupils were interviewed on each topic and asked how they would attempt to solve the problems which later appeared on the written test. From these replies, methods used by children (both correct and incorrect) were discovered. The description attached to the groups of items which formed the levels in the hierarchy were in some cases based on the methods the children used.

Each paper was given to children of three different ages; at each age level the sample was representative of the IQ distribution in the British child population and taken from at least six schools. The results of the testing are therefore thought to be a good indication of the different depths of understanding throughout the secondary school population (age 12-16).

The formation of a hierarchy in each topic (e.g. Ratio, Algebra, Fractions) was based on the grouping of items at a particular facility level using the homogeneity coefficient ϕ.[1] For items to form a group, they had to be linked (at an acceptable level) with harder and easier items and with others of approximately the same facility. The separation of one group from the next hardest was based mainly on the existence of a facility gap but also on the mathematical coherence of the items. When items on a particular test had been formed into groups these were designated 'levels' and each child in the sample tested was assigned to a level of understanding on the basis of his performance. A child was assigned to level 2 for example if he had passed about 2/3 of the items in each of level 1 and level 2. The levels were tested for scalability by using a Guttman scalogram analysis and if the percentage of children who had error type responses (i.e. success on a harder group did not mean success on all easier groups) was above 7 per cent, the chain of groups was reconsidered.

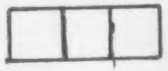
The hierarchy in each topic was found first and then a comparison between the performance of the different age groups was made. It was soon apparent that although there were more of the oldest subjects (15-16) at the highest level of attainment, each year group contained children at every level of understanding. This was further confirmed by the results of a longitudinal survey (over three years) carried out in three topics. The movement over the two years in ratio understanding, for example, was more often one level than two. The longitudinal sample was chosen to contain equal numbers in four IQ groups and the level of attainment both at age 13 and 15+ was closely allied to the IQ of the child. For example in the group IQ $\geqslant$ 110 there was no 13 year old child who could not cope with the easiest questions on the ratio paper whereas in the group IQ $\leqslant$ 89, 33 per cent could not cope with the easiest questions in 1976 and no child had reached the hardest level by 1978 (only one had reached the penultimate level). There appeared to be no sudden understanding of ratio and proportion which made the older children automatically more competent than the younger.

In order to be able to give information to teachers it was hoped that levels of understanding could be matched topic to topic. One problem was that there were not groups of items at the same facility level in each topic, for example Level 2 Graphs might contain items with facilities between 70 and 55 per cent and there were no ratio items of comparable facility. An approximate match when perhaps two levels in one topic corresponded to one in another was attempted. Additional information from the testing of subsamples of children who had completed two test papers was available but the problem of gaps in facility still applied. Every child tested had attempted two test papers but sometimes the number of children doing a particular pair of tests was small (e.g. 50 from four schools). A gamma coefficient, (which quantifies the probability that if Child A is ranked higher than B on say the ratio test, he is also ranked higher than B on a second test) could be found for each subsample however. The majority of the gamma coefficients between pairs of tests were in the range .70 to .85.

Some topics tested required the use of mathematical symbols and conventions before the child could start to deal with problems, e.g. fractions. Admittedly little secondary school mathematics can be accomplished without the use of integers and operations upon them but if one regards these as the basic vocabulary of mathematics, Algebra and Fractions require a new super vocabulary which assumes the first.

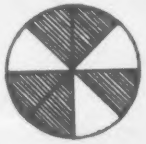
If we look at the items in the easiest levels in each topic (some items on the test papers were omitted since their correlation with other items was low) we can illustrate the need for conventions. Taking Stage 1 as all groups of items having a lower level of about 80 per cent success in the 15+ sample we find the comparable lower level is 75+ per cent for the 14+ children and 70 per cent for the 13+ children. The upper limits depend somewhat on the individual topics but would generally be about 95 per cent. The Fractions questions at this stage are of the type:-

1) Shade two thirds



or

2) What fraction is shaded



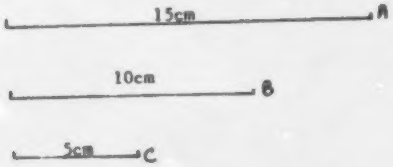
or

3) In a bakers shop 3/8 of the flour is used for bread and 2/8 of the flour is used for cakes. What fraction of the flour has been used? ....

These items test the meaning of fractional notation and the first convention for addition. The comparable questions in ratio and proportion deal with the ratio 2:1 in a whole number setting. The questions need a knowledge of positive integers but no other specialised vocabulary e.g.

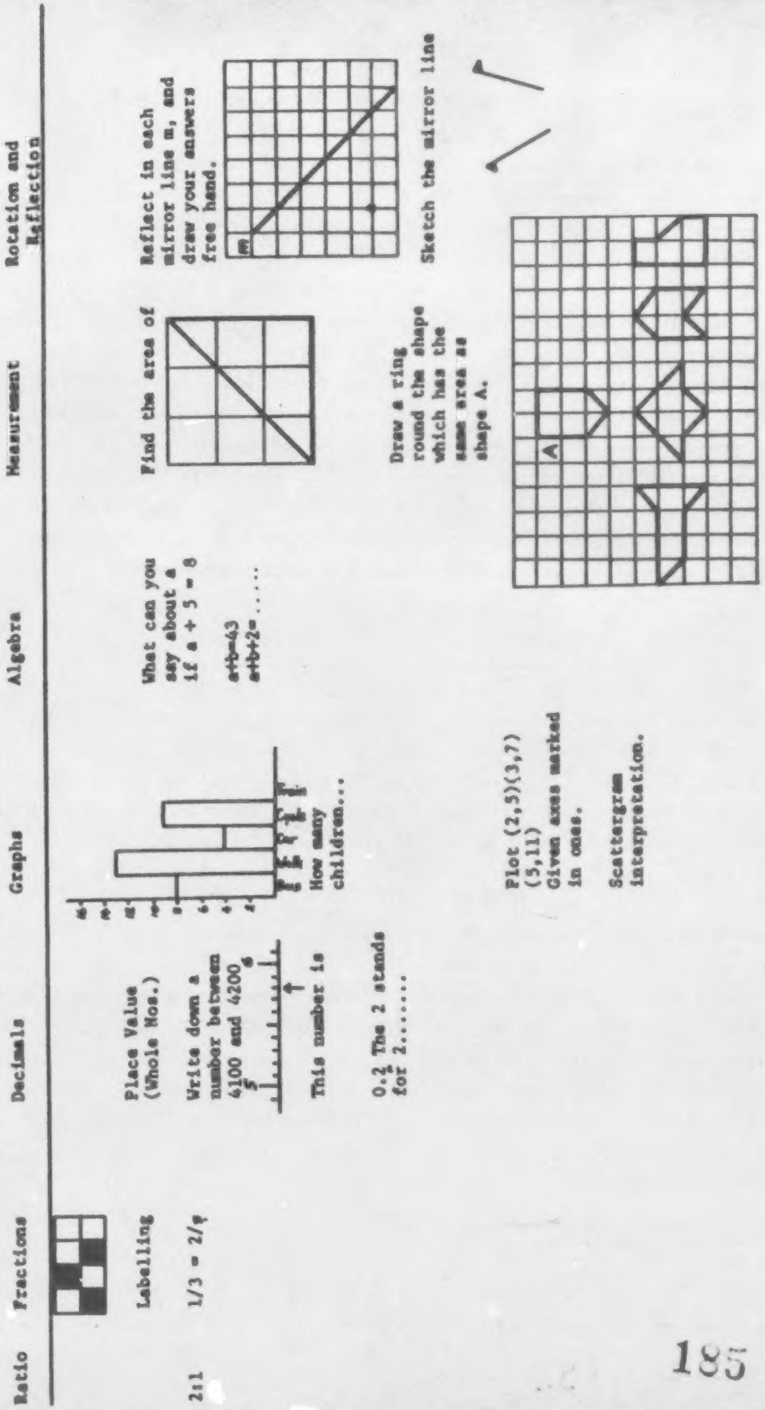
The eels are fed sprats, the number depending on their length.

If C is fed two sprats, how many sprats should B and A be fed to match?


15cm
A
10cm
B
5cm
C

Stage I descriptions of seven tests appear in diagram 1.

Diagram 1. Stage One Levels in Seven Mathematics Topics

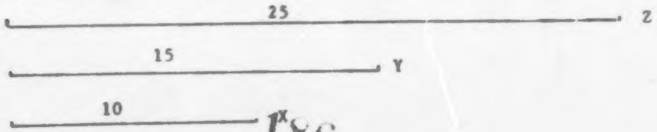

Ratio
Fractions
Decimals
Graphs
Algebra
Measurement
Rotation and Reflection
Labelling
2:1
1/3 = 2/6
Place Value (Whole Nos.)
Write down a number between 4100 and 4200
5
6
This number is
0.2 The 2 stands for 2.......
How many children...
Plot (2,5)(3,7) (5,11) Given axes marked in ones.
Scattergram interpretation.
What can you say about a if a + 5 = 8
a+b=43
a+b+2= .....
Find the area of
Draw a ring round the shape which has the same area as shape A.
A
Reflect in each mirror line m, and draw your answers free hand.
m
Sketch the mirror line

The items in Ratio, Graphs and Measurement show an application of the conventions used in those topics. In Ratio, doubling and trebling are required but no technical language is used. Pictorial representation is shown in the easiest level of Graphs together with the plotting of whole number coordinates. In Measurement the items can be solved by counting whole and half squares. Decimals, Fractions, Algebra and Transformation Geometry items at this level tend to be testing a simple knowledge of the language specific to the topic. This may not be surprising but the fact that nearly thirty per cent of our child population (even at age 15) can go no further i.e. cannot apply this language demonstrates the difficulty children experience in applying new mathematical terminology. The gap between the ability to deal with whole numbers and the ability to deal with fractions and decimals is wide. It is not a case of "he can manage whole number operations, fractional operations are a logical next step". The notation of fractions and the operation of addition are introduced in most British primary schools at about age nine or ten, it would seem that a case could be made for a postponement.

Comparison between the Hierarchies of Ratio and Fractions

In diagram 2 a description of each level in the hierarchies of Ratio and Fractions is given in terms of the most 'naive' method of solution used by children when interviewed e.g. repeated addition is considered more naive than multiplication. The Fraction items were in problem form, a discussion of the purely computational aspect of Fractions appears later. The Fraction paper appeared in two forms, that for the younger children omitted items requiring multiplication and division so this discussion concerns the sample of 14-15 year olds.

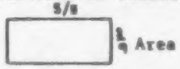
The hardest Ratio item required first the recognition of the ratio 5:3 and then its application. The repetition method (take it once, take it again, take a half and add) could not be applied in this case without a fraction other than a half being used so multiplication by a fraction had to be employed. This was at about the same level of difficulty as the area of a rectangle when the two dimensions were 2/9 and 5/8. i.e. requiring the multiplication of two fractions. The items in level 3 ratio included eel questions such as:-
Three eels X, Y and Z are fed with fishfingers, the length of the fishfinger depending on the length of the eel.



If Z has a fishfinger 10cm long, how long should the fishfingers given to X and Y be?

The repetition method is no longer easy to apply and can be used only if the rate per 5cm is found first.

Diagram 2

Ratio n=2257			Fractions 3, 4 n=523
STAGE 1			
Doubling, halving No rate i.e. amount per person not needed	Level 1	Level 1	Labelling part of a whole Equivalent fractions where doubling is involved 1/3=2/? Addition same denominator
75-80%			
STAGE 2		Level 2	Equivalent fractions not of obvious doubling form e.g. "A relay race is run in stages of 1/8 Km each. Each runner runs one stage. How many runners would be required to run a total distance of 3/4Km?
52-58%			
STAGE 3			
Obtains answer by repetition.	Level 2		
		Level 3	Double complexity of equivalence e.g. 2/7=4/14=10/?
Operation on a fraction e.g. 1/2(1/2)+1/2(1/2 of 1/2) Converse of repetition (diminution instead of enlargement) or rate must be found and then repetition take place.	Level 3		
22-30%			
STAGE 4			
Ratio is viewed as a multiplicative not additive operation ratio itself is 3:2 or 5:3	Level 4	Level 4	Multiplication of fractions e.g. 
Error rate 3.7%			Error rate 2.9%

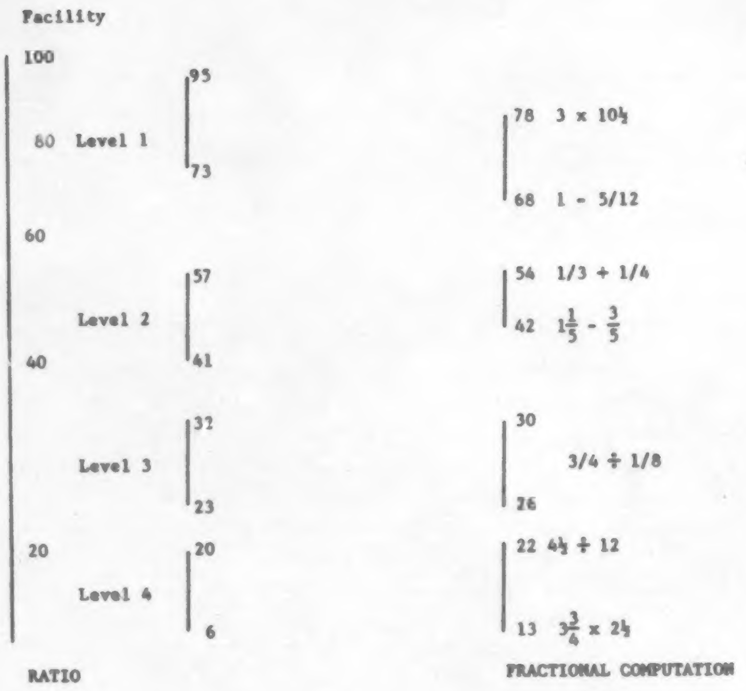
The comparable fraction questions are concerned with equivalent fractions when the conversion from one form to another cannot be achieved by one multiplication. There were no ratio items equal in facility to level 2 fractions, which involved straightforward equivalent fractions or the application of well known equivalents such as 3/4, 6/8.

Sixty-eight children in the 14-15 age group had attempted both the Fraction and Ratio test papers. A matching of their performance on each of the papers is shown below.

		Fractions					
		0	1	2	3	4	ROW TOTAL
	0	4	1	0	0	0	5
	1	6	11	10	4	1	32
Ratio	2	0	1	7	12	2	22
	3	0	0	0	6	1	7
	4	0	0	0	1	1	2
COLUMN TOTAL		10	13	17	23	5	68

It can be seen that a performance at two extremes did not occur but that a level 1 ratio performance could be matched by levels 1 or 2 in Fractions, recall that there was no ratio level parallel in facility to level 2 Fractions.

The same children were also given a series of fractional computations. The comparison between ratio facilities and the computations were as follows:-



The gamma coefficient was .6197, i.e. considerably lower than when the comparison was between the same Ratio levels and the levels obtained from groups of the Fraction problems. This tends to confirm the findings of Abramowitz (2) who stated "....skill tests of facility with fractions load on a different factor than tasks involving proportionality".

Many children found the problems involving fractions easier than corresponding computations highlighting the fact that they did not use the taught algorithms when they had available their own successful (even if naive) methods.

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The Understanding of Some Algebraic Concepts at the Secondary Level

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Introduction

Reporting on the state of research at the Third International Congress on Mathematical Education (Karlsruhe,1976), Heinrich Bauersfeld concluded that a general theory of the learning processes involved in mathematics could not be expected to materialize in the near future. He felt that before trying to generate theories on concept formation, it was necessary to specialize in the investigation of how specific concepts are learned. Such investigations needed to distinguish between the structure of the mathematical discipline, the "matter meant", how in turn this is shaped by the teacher, the "matter taught", and finally, how it is interpreted by the student, the "matter learned". These studies had to be preceded by broad descriptions of individual acquisition and understanding processes (Bauersfeld,1977).

The research reported in this paper reflects in many ways Bauersfeld's recommendations. The scope of this study is limited to the construction of meaning for first degree equations in one unknown and for linear equations in two variables. These investigations aim at uncovering learning processes by analysing individual acquisition and understanding. The analysis is based on case studies involved in two teaching experiments.

This article is divided into four parts:

1. The first part summarizes some results obtained by Wagner (1977) and by Kerslake (1977)
2. The second part deals with the theoretical and methodological models used in this research.
3. The third part describes the two teaching experiments, the first one dealing with the concept of first degree equations in one unknown and the second one dealing with the concept of linear equations in two variables
4. The last part discusses some results and attempts to reach some conclusions.

1. Two recent studies

Many studies on the learning of algebra have investigated the acquisition of algebraic skills but have ignored the conceptual framework which must subtend these skills. The research reported in this paper pertains to the construction of such concepts. It tackles the problem of constructing meaning for first degree equations in one unknown and for linear equations in two variables.

Two recent studies revealing the lack of understanding of algebra at the secondary level are of particular relevance. The work of Sigrid Wagner (1977) and of Daphne Kerslake (1977) goes beyond the mere testing of skills and deals with cognitive obstacles at the conceptual level. These researchers have introduced innovative and pedagogically valuable questions which can uncover the student's grasp of these concepts. Some of these questions have been used in the teaching experiments and in the post-tests reported in the main part of this presentation.

1.1 Wagner's "conservation of equation and of function"

In a study carried out in the United States, Sigrid Wagner has introduced two very innovative concepts which she called "conservation of equation" and "conservation of function". She asked 72 students aged 12, 14 and 17 whether the equations

$$7n + 22 = 109 \quad \text{and} \quad 7w + 22 = 109$$

would yield the same solution or not. The answers varied widely from "the solution of the second equation is greater than the first one because w comes later than n in the alphabet" or "can't tell until both equations are solved" to "of course, the solution is the same".

A student who responded that both solutions would necessarily be the same was said to "conserve equation" whereas a student who believed that either w or n would be larger was classified as a "non-conserver".

For the function task, the student was first shown a complete chart of ordered pairs with columns headed by X and Z. The heading Z was then changed to Y and one number was deleted from the chart. The student was asked to supply the missing number.

NOTICE THAT WHEN
X = 12 , Z = 27

X	Z
2	7
3	9
5	13
8	19
12	27
14	31

WHAT WOULD Y BE EQUAL TO
WHEN X = 12 ?

X	Y
2	7
3	9
5	13
8	19
12	
14	31

If the student supplied the original number, he was said to "conserve function". If he supplied a different number, he was classified as a "non-conserver".

Here are some of Wagner's results : Of the 12-year-olds, about 50% conserved equation but only 20% conserved function; of the 14-year-olds, over 80% conserved equation and 50% conserved function. The performance of the 17-year-olds in this study was not significantly different from that of the 14-year-olds, probably because a greater number of below-average students were included in the 17-year-old group than with the other two groups.

Wagner's research brings out clearly some of the cognitive problems which can be attributed to the symbolic nature of algebra. Her results are somewhat surprising since even in elementary school, children start using letters as numerical symbols.

That 20% of the older students did not conserve equation motivated the development of a novel construction, reported in the first teaching experiment. The fact that the concept of equation seems easier to grasp than the concept of function (at least when expressed in tabular form) provided some justification for avoiding a functional approach in the construction of meaning for linear equations in two variables.

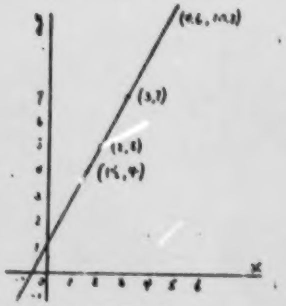
1.2 Kerslake's "understanding of graphs".

In constructing a test on graphs, Kerslake attempted to devise questions which would "test understanding of the principles involved". A total of 17 questions was produced and administered to 600 pupils in four secondary schools in Great Britain. The four questions presented below are the ones which have been used as part of the test and post-test in the second teaching experiment.

Question 1. The first question was designed to test the pupils' understanding of continuity and their ability to view the straight line as an infinite set of points. Given a few ordered pairs, (2,5),(3,7), etc., they had to plot the corresponding points and draw the line. They were asked :

1a : How many points do you think lie on the line altogether ?

1b : Are there any points on the line between the points (2,5) and (3,7)? If so, how many?

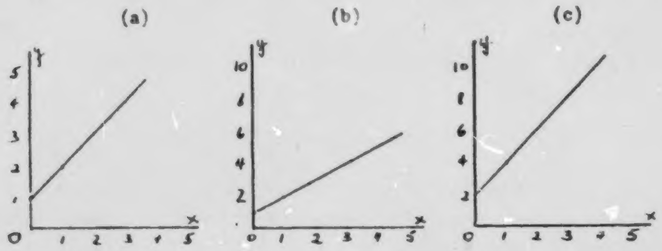


The answers ranged from "none", "one" or a small finite number to "hundreds" and "infinitely many". The percentage of students giving this last answer was :

	Year 2		Year 3		Year 4	
	Boys	Girls	Boys	Girls	Boys	Girls
Question 1a	8.5	2.0	12.1	2.9	46.5	32.5
Question 1b	6.4	0.0	10.1	1.9	33.7	25.4

Although a major increase occurs in the last year, the majority of these students do not view the straight line as an infinite set of points.

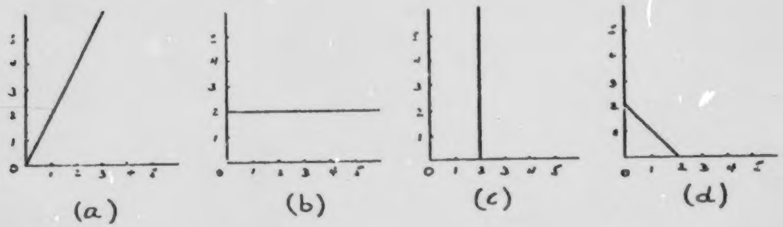
Question 2. The second question tests the responses of the pupils to a change of scale. They are asked "Two of the lines (a), (b) and (c) represent the same information. Which two?"



Percentage of pupils with correct response !

Year 2		Year 3		Year 4	
Boys	Girls	Boys	Girls	Boys	Girls
50.0	43.1	67.7	59.2	72.1	65.8

Question 3 :Which of the graphs below represents the line $y = 2x$? What are the equations of the other lines ?

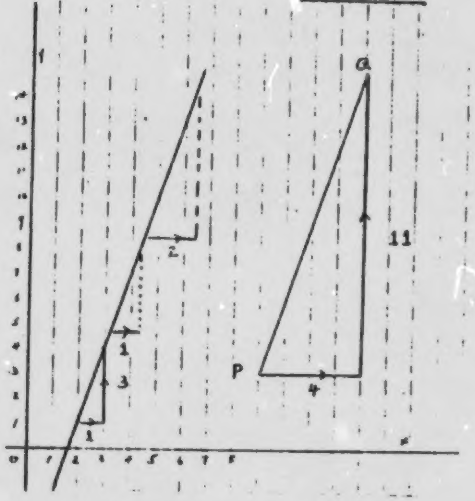


Percentages of correct responses :

	Year 2		Year 3		Year 4	
	Boys	Girls	Boys	Girls	Boys	Girls
(a)	13.8	21.6	35.4	19.4	36.0	36.8
(b)	3.2	4.9	8.1	10.7	26.7	22.8
(c)	1.1	5.9	9.1	8.7	27.9	22.8
(d)	0.0	1.0	3.0	2.9	14.0	8.8

The last question deals with the invariance of slope and the relationship between the slope and the direction of a straight line.

Question 4



What is the length of the dotted line ?

What is the length of the broken line ?

Is the line joining PQ parallel to the first line ?

Percentage of correct responses :

Year 2	Year 3	Year 4
10.2	10.4	37.7
24.5	35.0	38.0
1.5	1.5	9.0

Kerslake's results show that the understanding of the straight line and its equation is far from being attained by the majority of even 4th year students. But an examination of these questions indicates that different kinds of understandings are involved in these questions.

The notion of point and continuity are at this level purely intuitive and likely to depend on the development and growth of the child; the understanding aimed at by the second question involves the relation between graph and scale; the recognition of the graph of an equation and the generation of the equation of a given graph involve the understanding of a symbolic form; the last question brings in the intuitive understanding based on the perception of two lines which seem parallel, a false perception which would be corrected by noticing the different slopes.

This brief analysis shows the need to distinguish between different kinds of understandings. This has been attempted in the various models of understanding mathematics described in the following section.

2. Theoretical basis

As can be seen from Wagner's and Kerslake's data, many high school students find it difficult to understand the concept of equation. They do not seem to have developped any meaning for this notion. The study of such a pedagogical problem in concept formation necessitates a model which can help describe the various modes of understanding, a model dealing with its acquisition and, a methodology which allows the researcher to probe the learner's mind.

2.1 A model of understanding

In analysing understanding, Bruner (1960) found it useful to distinguish between intuitive thinking and analytic thinking. In 1976 Skemp described two modes of understanding, instrumental understanding and relational understanding. A new model was suggested a year later by Byers and Herscovics (1977) which incorporated the previous two and furthermore distinguished between mathematical content and mathematical form. It described the following four modes:

instrumental understanding is evidenced by the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works

relational understanding is evidenced by the ability to deduce specific rules from more general mathematical relationships

intuitive understanding is evidenced by the ability to solve a problem without prior analysis of the problem

formal understanding is evidenced by the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these idead into chains of logical reasoning

The first three modes are described in terms of problem-solving and emphasize the use and derivation of rules. However, in the framework of concept formation these descriptions seem somewhat inadequate, for if problem-solving can be associated with the restructuring of existing knowledge (in the Gestaltist sense), concept formation on the other hand is more concerned with the expansion of this existing knowledge,i.e. the construction of new concepts. In the previous definitions the use and derivation of rules seem to be out of context, an end in itself; whereas when viewed in terms of concept formation, rules and procedures can become the means to link up different mathematical ideas and representations. These considerations warrant a new model of understanding better suited to situations involving the construction of new concepts.

Instrumental understanding was described by Skemp (1976) as "rules without reason". This definition needs to be expanded, for rules can become the means to operare on and to justify some concepts. For instance, a student may feel intuitively that there must be infinitely many rationals betwean 1/3 and 1/2 . But when he learns to express these fractions as 2/6 and 3/6 or better yet as 2000/6000 and 3000/6000, he finds an instant justification for his answer. Thus, instrumental understanding must be redefined to include the use of algorithms and procedures as instruments operating on and linking up various concepts.

Relational understanding has been described as the ability to deduce specific rules from more general mathematical relationships. This definition is not adequate in concept formation,for in such a situation the term "relational" is better used to describe the understanding of a student who can establish links and perceive relations between different mathematical concepts and representations. This description of relational understanding implies a necessary interplay with instrumental understanding for it seems impossible to secure links and relations without the needed rules and procedures. Of course, a pupil focusing on the rules may be oblivious of the context and thus be limited to instrumental understanding. On the other hand, a student may very well remember a mathematical relationship and forget the procedures involved in the initial construction. He would then end up with relational understanding but without instrumental understanding.

Intuitive understanding acquires a different meaning in concept formation, for it can be associated with the various representations of a concept. Bruner (1966) identifies three modes of representation (enactive, iconic, symbolic) which he clearly relates to modes of understanding: "A young student may come to a clear understanding of factoring and prime numbers by visualizing them in terms of the way in which quantities of beans can or cannot be laid out in columns and rows."(Bruner,1973). Thus enactive and iconic representations can be considered as more intuitive than symbolic ones and lead the learner to an intuitive understanding. Bruner's different representations are primarily related to the maturation of younger children.

Even at a higher level, one can consider different mathematical representations of a given concept and compare them in terms of being more intuitive or formal. For instance, a numerical table may convey the concept of a function more intuitively than a graph if the student is not familiar with the convention of the Cartesian plane.Hence one can describe the understanding resulting from a non-formal presentation as being more or less intuitive.

A third useful description of intuitive understanding relates it to the student's ability to think inductively, to generalise from a few examples, to recognize a numerical pattern. Of course, it is impossible to dissociate this from his analytic thinking for presumably, the latter is almost immediately brought to bear on his "guesses" in the form of verification.

Formal understanding as previously defined consists essentially of two parts. The first part, involving the ability to connect mathematical symbolism and notation with relevant mathematical ideas, seems very appropriate to concept formation for many of the concepts taught at the elementary and secondary level are mathematical forms (e.g.equations) which are new to the students and for which they have to construct meaning.

The second part deals with the combination of these ideas into chains of logical reasoning. If these are viewed as a sequence of steps in a formal proof, then they can be interpreted as formal understanding. However, if the context is concept formation, the linkage of the mathematical ideas seems to be better described in terms of relational understanding.

The four modes of understanding that are suggested here are obviously not independent and probably never exist in a pure state. Nevertheless, this model has been useful in the two projects reported in this paper, for it provided a frame of reference and the vocabulary needed to describe the pedagogical potential of each representation and the corresponding understanding that ensued.All four modes were integrated in the teaching-learning model described below.

2.2 Didactic reversal, a teaching-learning model

Although the models of understanding provide the language needed to describe it, they do not cope with the question of how understanding can be acquired by the learner.

A new concept can be introduced by connecting it to a simpler one or to an equivalent concept known to the student.To deal with a new concept by expanding a simpler one involves concept formation. For example, algebra in one variable can be seen as generalised arithmetic. On the other hand, to connect a new concept with an equivalent one can be viewed as a problem of representation. For instance, algebra in two variables can be considered as a representation of plane geometry.

How can these connections between the new material and the cognition of the learner be established ? Too often, the teacher fails to realize that he has the choice of two possible approaches. He can start with the new topic and transform it to reach the student's cognition or, on the other hand, he can start from the student's cognition and transform it to reach the new topic. What are the possible consequences of each approach ?

Using the first approach, which could be called a formalistic one, if the new material is not yet understood, the transformations performed on it may conceal its significance and create a problem of accommodation which may be too large for the student to accomplish.

Using the second approach, which can be called constructive, transformations performed on the student's cognition in order to reach the new topic allow him to build meaning and thus represent a process of assimilation. This process of assimilation can lead to the desired change in the learner's schemas for "effective assimilation has as its counterpart a more or less effective accommodation"(Piaget and Inhelder, 1947).

These general considerations have been translated into practice. Piaget's theory of equilibration has been adapted in a teaching-learning model, didactic reversal (Herscovics,1979),which has been applied to some algebraic concepts.Its main features are:

1. algebra can be viewed as a representation of arithmetic and geometric ideas
2. the teaching of algebra should aim at establishing equivalences between various representations
3. algebra is a mathematical form new to the student and he must first be guided in constructing meaning for it
4. to construct meaning for the algebraic form, the student must build on his knowledge of arithmetic and geometry
5. only after meaning has been constructed for the algebraic form should the problem be reversed and the student led to find the arithmetic and geometric representations, for then the necessary operations will not have to be performed on meaningless symbols
6. this process should be applied globally as well as to all the intermediary concepts.

Didactic reversal has been used to develop two experimental teaching outlines. To prepare such outlines, one has to start by looking at those arithmetic and geometric ideas,existing in the student's cognition,which can be expanded or are equivalent to the algebraic concepts to be taught. From such a starting point one needs to determine those intermediary concepts which will be necessary in the construction. Each intermediate concept has to be analysed in terms of content and form. Is it an expansion of a known mathematical idea or is it a new representation ? It is in this process of analysis that the ability to identify modes of understanding is extremely useful. For if accommodation is to be achieved through assimilation, initial communications between teacher and pupil have to be at the intuitive and relational level. This allows for continuity in the expansion of mathematical content before this content is frozen into symbols which might still be meaningless.

Despite the care and preparation provided in the design of the teaching outlines, these are but reflections of "the matter meant" and "the matter taught". Only by experimenting them on students can one hope to find "the matter learned". This, however, requires a methodology which allows the investigator to uncover the student's thinking.

2.3 Methodology

Nothing seems to divide researchers better than a discussion of methodology. Some psychometricians feel that all clinical research is merely anecdotal, not scientific. Some investigators using clinical interviews believe that all psychometric work is a self-deluding attempt at using hard data on a soft science. Of course, both attitudes are dogmatic and counterproductive. The correctness of a methodology can only be judged on the basis of whether or not it is appropriate for a given investigation. And it is obvious that both methodologies are essential: clinical work seems best suited to uncover the thinking patterns and understandings of the students, psychometric research can determine how widespread these thinking patterns are.

The methodology used in the research reported here was introduced in North America in the early seventies and is known as the "Russian teaching experiment". As pointed out by Kilpatrick and Wirszup,(1969),it reflects the philosophy of Soviet psychologists who ascribe a leading role to instruction in the development of the child. According to Menchinskaya (1969),Soviet instructional psychology concerns itself with the characteristics of mental activity during instruction and with the principles of the learning process itself. In order to study the changes in mental activity, Soviet didacticians (such as Vigotsky in the early twenties) devised the "teaching experiment". It exists in many forms such as the "experiment of instruction" and the "experiment of assessment" . Experiments of instruction involve a preliminary phase in which the initial information , the abilities and skills needed to master the new material,are determined and ordered in some hierarchy. The instruction which follows can be applied to regular classes, small groups of children, or individual students.

Kantowski (1979) points out that most studies deal with some aspect of the school curriculum and the data is gathered from a sampling of strong, average and weak students (selected with the aid of the classroom teacher) in a clinical setting by recording the verbal protocols. The same children are studied for periods of time which can range from six weeks to the academic year.

These guidelines were followed in the two teaching experiments presented in this paper. The students were selected by their teachers on the basis of their schoolwork and represented a wide range of abilities. Each pupil followed an experimental teaching plan and each interview was tape-recorded and transcribed. The protocols were used in conjunction with the written work to analyse the student's thinking and the results were compared for individual differences and common thinking patterns.

The first experiment, in collaboration with C.Kieran, was on the teaching of first degree equations in one unknown. Each of six children in grades 7 and 8 (aged 12 to 14) was interviewed five times with the exception of one child who required eight sessions. A post-test was given six weeks after the last meeting.

The second experiment was on the teaching of linear equations in two variables. Three grade 9 students, aged 15, were selected (weak, average, strong) and each one was met nine times, except for the weak student who was met ten times. A post-test was administered three months after the last meeting.

3. The two teaching experiments

3.1 Teaching first degree equations in one unknown

Most presentations start by introducing equations such as $3x + 5 = 17$. Students are taught to manipulate them in order to find a solution, $x = 4$, which they substitute for verification and obtain $3 \cdot 4 + 5 = 17$ an arithmetic identity.

Applying didactic reversal to this topic meant to start with arithmetic and to lead the student in the construction of equations.

Initially, these junior high school students' concept of an equality was similar to that found by Herbert Ginsburg (1977) in younger children, that is, an operation on one side and the result on the other, as in $2 \cdot 3 = 6$. In order to construct meaning for non-trivial equations, it was decided to expand the students' notion of equality to include multiple operations on both side. When asked to use the equality symbol with one operation on each side many students responded with examples of commutativity such as $2 \cdot 3 = 3 \cdot 2$ However they had no difficulty in constructing examples with different operations on each side but ignored the use of brackets and the conventional order of operations (see C.Kieran, Children's Operational Thinking within the Context of Bracketing and the Order of Operations, in these Proceedings).

The concept of an equation could now be introduced in small steps leading up to the new mathematical form. One of the student's arithmetic identities was chosen and one of its numbers was covered up with a finger. Thus an equation was defined as

an arithmetic identity with a hidden number

Defining equations in such a simple way allowed the student to grasp the concept prior to being confronted with the new form. He could then move on to a more practical representation and replace the hiding finger with the familiar empty box. This was an intermediate step and the box was then replaced with any letter chosen by the student. The concept and terminology of an unknown could now be assimilated. It should be noted that this construction led the student to the concept of the solution of an equation but not to the solution processes. The solution of an equation was perceived by the students as being the hidden number.

Kieran has found that arithmetic identities can also be used to justify the operations used in solving equations. Most teachers use the idea of equilibrium embodied in a scale to explain "doing the same thing on both sides of an equation". And, as long as they limit themselves to addition and subtraction of natural numbers, the scale is perfectly adequate. But it may be difficult for the student to perceive the other operations concretely on the scale. Such restrictions do not exist in the use of arithmetic identities. Children learn quite readily the rule "do the same thing on both sides", for otherwise they lose their arithmetic identity.

For small numbers, students have no difficulty in solving simple equations involving one or two operations. In fact, their "guessing" of the answer is nothing but an extension of mental arithmetic. It is when terms in the unknown appear on both sides that mental arithmetic ceases to work and algebraic methods are needed. For such equations, students were taught to gather like terms , thus reducing the equations to easy-to-solve types.

The protocols of the interviews dealing with the construction of meaning have been analysed. Those dealing with solution processes are still being studied. and only first impressions can be reported. It is rather surprising to find that although the students had developed meaning for the algebraic form and could justify each operation, they did not seem to develop an overview of the various possible strategies needed in the solution process. They seemed to find it difficult to know where to start and how to link up the various steps required to solve an equation. In the post-test given six weeks later, they had retained the meaning of equation and could justify the algebraic operations but could not solve an equation until they were reminded of the process by seeing a written-out solution.

3.2 Teaching linear equations in two variables

Standard presentations of this topic start with linear equations such as $y = 2x + 3$. Students are then asked to find a few ordered pairs, plot the corresponding points in the plane and join them by a straight line. They have gone from an algebraic form to a geometric representation.

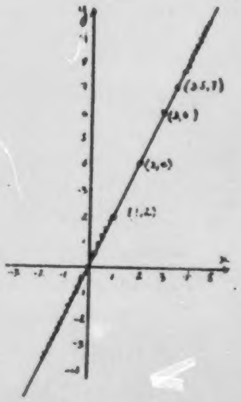
Applying didactic reversal meant to start with geometry. None of the three selected students (Mr.Strong,Average,Weak) had seen or worked with linear equations and their graphs. In fact they had to be taught the construction of the Cartesian plane. When asked how to describe the position of a point in the plane, the three students handled this problem intuitively by giving the coordinates. The need for ordering them became evident as soon as they were interchanged. Having identified points in the plane with ordered pairs, the inverse identification was spontaneous. The students were then asked how to describe the coordinates of a variable point in the plane. They agreed that an ordered pair of numbers could not be used since such a pair determined a fixed point, hence the need to use an ordered pair of letters (x,y).

In order to construct the equation of a straight line, the pupil must realize that the linear equation merely describes the relationship between the coordinates of any point on the given line. This creates a pedagogical problem since,as Kerslake's results indicate, the majority of students do not think of a straight line. as an infinite set of points. However, Piaget and Inhelder (1947) have shown that starting at the age of 11 or 12, Swiss children can envisage the possibility of drawing an infinite number of points on a straight line. Such was not the case with the three 15-year-olds in this experiment. They had not abstracted the notion of point and were conceiving it as a dot or a small circle. Hence, according to them, only a finite number of points could be drawn on a finite line segment. Nevertheless, this proved to be sufficient since in an initial construction, the student merely had to "cover" the straight line with points.

The two constructions of linear equations that were attempted with the students are now described. The first one is called "intuitive" since it is based on pattern recognition and guessing, whereas the second construction is called "relational" since it is based on relating the concept of slope with that of direction.

3.2.1 Intuitive construction

Students were given a straight line which they were to cover with points. After finding a few coordinates they were asked if they could see a pattern or relationship between the first and second coordinates. All three recognized that "the first one is half the second" or "the second is double the first" and were requested to write down in words the rule they had just verbalised. The variable point (x,y) was now introduced, but restricted to any position on the straight line, and using its coordinates they were able to express with very little help their rule " $2 \cdot x = y$ " or " $x = \frac{1}{2} \cdot y$ "

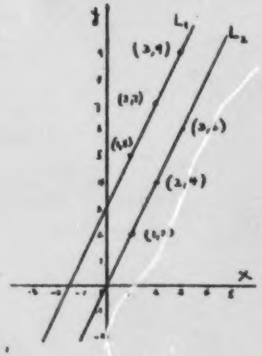


In order to establish some contrast, students were to verify if the coordinates of points off the line would balance their equation. They had no problem in substituting these numbers and coming to the expected conclusion.

Thus it is quite surprising that when asked to use their equation to generate ordered pairs and guess if these would correspond to points on the line, none of the three students were able to understand the question. Although they had derived their equation from ordered pairs, they could not use it to generate more ordered pairs. Indeed, they had to be shown how to do this instrumentally by substituting a value for x and computing the corresponding value for y. Even then, they did not use these numbers spontaneously as ordered pairs. Mr.Average thought that these ordered pairs would not give him points falling on the straight line.

It was also surprising to find that after deriving the first equation, two of the three students (Strong and Average) thought that all straight lines through the origin would have the same equation. Of course, by repeating the process with other straight lines they concluded that each one had its own unique equation. At the conceptual level, no significant difference could be found among the pupils. Mr.Weak had difficulties with fractions and negative numbers and in fact, throughout this experiment, he was plagued by his weakness in arithmetic.

Still at the level of intuitive construction, the students were asked to guess the equation of a straight line, L_1, with a non-zero y-intercept. By using a protractor, a parallel line, L_2, was drawn through the origin and points with identical abscissas were identified on each line. The y-coordinates could then be obtained in two steps : L_2 yielded $y = 2x$, and a comparison with L_1 provided the additive component, thus resulting in $y = 2x + 3$.



All students found this extremely difficult and even after one hour of instruction could not derive such equations. Nevertheless, they developed some formal understanding for they could now recognize that straight lines not crossing the origin had equations which involved a two step rule.

The aim of this first construction was to provide linear equations with some readily accessible meaning. To the question "what is meant by the equation of a straight line?", the expected answer was "it's a rule which describes the relationship between the coordinates of points on the line". However, repeated attempts to elicit this response from the students failed even when they were fed the answer. Inevitably, to the question "what is?" they would respond with "how to" and describe the process by which they found the equation. This confirms Colette Laborde's findings (1978) regarding students' preference for the dynamic aspect of mathematics over the static one.

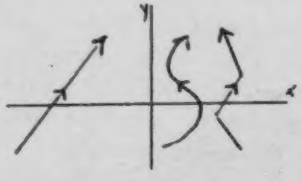
The intuitive construction remains an essential first step; however, because of its obvious limitations, it needs to be complemented by an alternate approach.

3.2.2 Relational construction

The relational construction of a linear equation was based on a property which characterizes the straight line namely, the invariance of its slope.

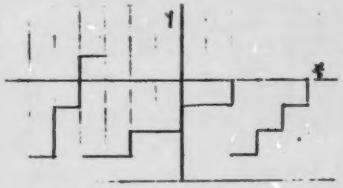
Invariance of direction

When presented with a straight line in the plane, the students didn't think it had a direction. It could have one "if you put an arrowhead at the end". They readily agreed that the direction didn't change if the arrowhead was shifted anywhere down the line and that this property did not hold for curves and broken lines. The converse was not so obvious. "Is a line whose direction never changes necessarily a straight line?". After some reflection the answer was "yes" and the pupils became aware that this was a characterizing property ("only straight lines could have it").



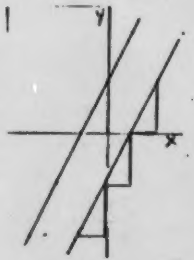
Slope

The concept of slope was introduced very concretely by using staircases (Petsu,1977). After defining *tread* and *riser* as the horizontal and vertical parts of a step, students were asked to draw a staircase *going up* whose riser was twice the tread. Since one could go up and down the same staircase, the need for a convention became obvious and the standard one based on the directions of the axes was adopted. By drawing two other staircases "going up" whose riser is half or equal to the tread, students were able to compare the slopes in terms of steepness. They realised that the slope was related to the riser and the tread and attempted some definitions expressed as equations which were easily converted to the ratio: riser/tread. A discussion of staircases "going down" led to the convention of negative slopes but even extensive practice, measuring the slopes of given staircases, did not avoid mistakes regarding their sign. It is only when the problem was reversed, when students had to draw staircases corresponding to given slopes, that the convention was mastered.



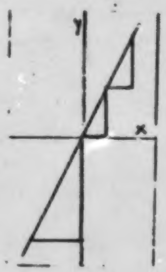
Slope of a straight line

When asked to define the slope of a given straight line, the three pupils knew that it was related to their previous work but didn't quite see how. Attaching a staircase to the line made it self-evident. Again, after measuring the slope of many lines, the problem was reversed and students had to draw lines with given slopes by first drawing the corresponding staircases. These constructions allowed them to find many lines with the same slope. They noticed at once that these were parallel lines and conversely, that parallel lines had the same slope. Thus direction and slope were tied together, slope being perceived as a measure of direction.



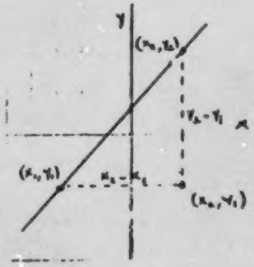
Invariance

It didn't take long before the students spontaneously ceased to draw staircases and limited themselves to one single step. This led to a discussion of the invariance of the slope. Since none of them had done any geometry, it could not be proved formally and had to be established inductively. They readily agreed that changing the position of the step along the straight line would not affect its slope. A request to vary the size of the step was not understood until a picture was drawn. The average and weak students needed to go through the actual evaluation in order to reach the desired conclusion. None of the three students could answer the converse (Is a line whose slope "never changes" necessarily a straight line?) until a curve and a broken line were drawn in front of them.



Formula for slope

The switch from a graphical determination of slope (counting squares) to a more formal one was achieved by pointing out that any two points on the line were sufficient to determine a step and that their coordinates could be used to measure the riser and the tread. Identifying the coordinates (x_2, y_1) of the corner facilitated this process. From a few numerical examples, the students were able to derive the formula for slope, $\frac{y_2 - y_1}{x_2 - x_1}$.



The order of the subtractions was discussed and the interchangeability of the two points was given an instrumental justification: $\frac{y_2 - y_1}{x_2 - x_1} \cdot \frac{-1}{-1} = \frac{y_1 - y_2}{x_1 - x_2}$.

The graphical method was so well entrenched that all three pupils showed some reluctance to use the formula. Hindsight indicates that the constant use of graph paper might have been counterproductive making it too easy to count squares.

It should be noted that it is at this formal level that some erratic behaviour became noticeable. The three students tended to invert the formula for slope, using tread over riser. Although Mr. Average knew very well that x written with a superscript 2 meant the square of x, he nevertheless choose to write $\frac{y^2 - y^1}{x^2 - x^1}$. Mr. Weak first wrote $\frac{y_3}{x_3}$ and then $\frac{y_2 , y_1}{x_2 , x_1}$ three days after having derived the correct formula.

Linear equations

The students were now presented with a straight line (the graph of $y = 2x + 3$) and asked to measure its slope. They were then required to draw any fixed point, (1,5), and a (different) variable point (x,y) on the line. Substituting the coordinates of these two points in the formula for slope they obtained $\frac{y - 5}{x - 1}$. Only Mr. Strong connected this spontaneously with the value of the slope to find the equation $\frac{y - 5}{x - 1} = 2$. The other two students had to be reminded that they had measured the slope and could use its value to write an equation. This was called the "rational" form of the equation.

From the rational form, it was but an easy step to obtain the point-slope form of the equation. Since x was standing for the abscissa of the variable point it could be thought of as a number, and hence multiplying both sides by (x - 1) gave the point-slope form $y - 5 = 2(x - 1)$.

The algebraic operations required to transform this last equation to the slope-intercept formula, $y = 2x + 3$, proved to be easy except for Mr. Weak whose manipulations were mechanical, without thinking. This final result allowed the students to increase their formal understanding, for they could now recognize in the equation the values of the slope and the y-intercept.

Having derived the slope-intercept formula, the following question was raised: How did we know that such an equation must must be that of a straight line? A reversal of the previous steps was suggested in order to obtain the rational form.

$y = 3x + 5 \qquad y - 5 = 3x \qquad \frac{y - 5}{x} = 3 \qquad \frac{y - 5}{x - 0} = 3$

This could be interpreted as follows: on the left a fixed point (0,5) and a variable point (x,y) have been used in the formula for slope. The equality indicates that the slope is constant. Which is the line with constant slope? The straight line. Mr. Strong seemed to be the only one to grasp this and responded smilingly with "I understand but it's really hard to explain".

It is interesting that when presented with a general linear equation such as $2x + 3y + 6 = 0$ the students did not recognize it as the equation of a straight line. Mr. Average said that "It couldn't be the equation of a straight line since x and y are not related by an equality". Of course, a few transformations were sufficient to change their mind.

Horizontals and verticals.

Tackling the equations of lines parallel to the axes, the pupils were asked to use their newly acquired methods to derive the equation of a horizontal line. The new form $y = k$, a constant, seemed to be somewhat unsettling and needed to be interpreted. It was enough to look at the restrictions on the coordinates of points on the horizontal to find the justification. After a little practice, they were able to write such equations on sight and transferred these skills to vertical lines. This transfer prevented them from being bothered by the fact that the slope of a vertical was undefined and thus could not be used to derive the equation.

Graphs

Having done all this work on the equation of straight lines, the reverse process, that of finding the graph of linear equations was not expected to cause any difficulty. However, when given the equation $y = -x + 4$ and told to find the straight line, the three students experienced the same problems observed at the beginning following the intuitive construction. They could not spontaneously use their equation to generate ordered pairs. When taught how to find solutions, by substitution, it became evident that the negative coefficient of x was compounding the problem which warranted re-writing the equation as $y = -(x) + 4$.

Even with the solutions in hand, the students did not immediately think of ordered pairs and had to be prodded into doing so. It took another couple of linear equations to firm up their concept of graph, which was then defined formally.

To verify if this concept could easily extend to non-linear equations, they were required to find the graph of $y = x^2$. Expecting a straight line, they initially wanted to limit themselves to two ordered pairs. Plotting a few more points, they soon realised that the graph was not going to be a straight line. Mr.Strong and Mr.Average connected their points on the parabola with a ruler whereas Mr.Weak joined them in a smooth curve.

The students were asked why they were joining these points. The expected response was "because the graph of an equation is the infinite set of points corresponding to the infinite set of ordered pairs that can be generated by the equation". Perhaps, as was the case with the definition of a linear equation, it was the *static* nature of the definition of graph which pupils found difficult. The best answer obtained was "to get a better picture".

Even if not verbalised, the concept of graph was easily generalised to non-linear equations. In fact, it seems that this concept might have been introduced immediately after the intuitive construction of linear equations, thereby providing the student with some contrast between linear and non-linear equations.

Test and Post-test

On the last meeting of the teaching experiment (end of June), the students were asked the four questions selected from the Kerslake test. No significant difference among them could be detected at the intuitive and relational levels of understanding. Only on the third question which involved a fair amount of formal understanding, the recognition and generation of linear equations, was there a marked difference among the students: Mr.Strong managed without any help, Mr.Average required some guidance, Mr.Weak had to be led step by step.

The post-test was administered three months later (end of September). The students had had two months of vacation and one month of class. That the topic of equations and graphs had not yet been introduced in class was verified with the pupils and their teachers.

The post-test results are somewhat surprising. At the end of three months, there was hardly any difference among the students at the level of formal understanding. Mr.Strong and Mr.Average seemed to have lost much of the formal understanding they had acquired during the teaching experiment. However, at the levels of intuitive and relational understanding, it appears that not only did the three students conserve their comprehension but that it actually *improved* over the summer. These observations are based on a comparison of the responses obtained in the June test with those of the September post-test:

- Recognizing the equivalence between linear equations and straight lines in the plane can be qualified as relational understanding at a global level. Such understanding was demonstrated by each one of the three pupils. In September, just as in June, they thought that each straight line had an equation and conversely, that the graph of each linear equation was a straight line.
- Viewing a linear equation as "the rule describing the relationship between the coordinates of points on a straight line" and perceiving the graph of an equation as "the points corresponding to the set of ordered pairs that can be generated by the equation" can be described as relational understanding at the local level. In the test as in the post-test, none of the three students could provide such static definitions of the two concepts but could explain and describe their constructions. In the case of linear equations they referred to the intuitive construction (finding points on the line and guessing the rule). In the case of graphs, they simply drew the graphs of $y = 3x$ and $y = x^2$. Whereas in May and June they had problems generating ordered pairs from their equations, this difficulty seems to have dissappeared in September without any further teaching.

- The ability to distinguish between linear and non-linear equations is certainly related to formal understanding (in the sense of mathematical form). Students were asked in the post-test which of the following equations was that of a straight line

$$y = 5 - x^2, \quad y = x^2 + 4, \quad y = 2x + 7, \quad y = 1/x$$

Only one student, Mr.Weak could not identify it.

- The first question of the Kerslake test dealt with the number of points on a line or between two given points. In September as in June, the three pupils answered "infinitely many".

- In answering the second question of Kerslake's test (Which two graphs represent the same information?) students had to overcome the misleading perceptual information and notice the change in scale. In June, Mr.Strong and Mr.Weak had initially made wrong guesses which they were able to correct by testing the coordinates of some points. However, in September there were no mistakes, the change of scale being noticed at once.

- An examination of the fourth question indicates that the concepts of slope and its invariance were well established. None of the three pupils made any mistake on the first two parts of this question.in the test or in the post-test. That they had linked slope and direction was evidenced by their answer to the third part of the question. Mr.Weak was the only one who thought, in June, that the two lines were parallel. In September there were no mistakes.

- The third question being the most difficult one requires a detailed analysis. Students were given four graphs and were first asked:

Which is the graph of $y = 2x$?

In June,Mr.Strong's and Mr.Weak's guesses were incorrect and they discovered their mistake by deriving three of the equations relationally. In September there were no mistakes, all students finding the correct answer by using the more intuitive method.

Horizontals and verticals

In June they could on sight write the equations of lines parallel to the axes. In September they all could observe and verbalize that the y-coordinate of the horizontal line was always equal to 2 but none of them could write down the equation spontaneously. Repeated requests for the equation received the same explanation. Only when asked to write down what they had just verbalised were they able to write the equation. The transfer to the vertical was immediate.

The equation of (d), ($y = -x + 2$)

In June, Mr.Strong derived the equation without any help, Mr. Average needed some guidance and Mr.Weak required so much help that his understanding of the relational construction was judged to be minimal. In September, although all students had retained the concept of slope and that of its invariance (as shown by question 4), none of them remembered the formula for slope and thus could not derive the equation. Repeated attempts to guess the rule connecting the coordinates of points on the line remained futile.

4. Discussion and conclusions

This section is devoted to an evaluation of the models and the methodology used in the two projects. The experimental results are discussed in terms of their pedagogical significance and their implications for further research.

The two fundamental ideas at the heart of the model of understanding -the identification of different modes and the distinction between content and form- have been extremely useful in the development of the teaching outlines and in the study of the students'thinking. The model helped in ordering the intermediary concepts and in their analysis for content and form. For each one of these concepts, intuitive modes of representation were found and links between them were established by the pupils. This has resulted in understandings which can be qualified as intuitive and relational.

It is only when these had been acquired that attempts were made to formalize them, thus leading to formal understanding. And it is precisely at this level that differences among the students were noticeable during the teaching experiments. If at the intuitive and relational levels no significant difference could be observed, in terms of formal understanding they could be ranked in an order corresponding exactly to their selection by their teachers. Of course, this raises the question regarding the kind of evaluation practiced in the schools, whether or not it is based solely on formal understanding. Also noteworthy is the fact that after a period of three months, in the post-test, the differences in formal understanding had almost vanished.

The acquisition of understanding has been considered in the context of Piaget's theory of equilibration. This theory was translated into a model, didactic reversal, which aimed at transforming the learning of equation from a problem of accommodation into a process of assimilation. This was achieved by using a constructivist approach, that is, by building on the students' existing knowledge of arithmetic and geometry. The expression "didactic reversal" has been used to describe both the model and the actual reversal of each initial construction.

In applying this model to linear equations, the objective of the intuitive construction was to build meaning for this new mathematical form. The inversion of this construction, leading to the graph of an equation, was deferred until after the relational construction. This postponement was probably a pedagogical mistake since many other concepts were introduced between the initial construction and its reversal. By introducing graphs immediately after the intuitive presentation of equation, some learning difficulties might have been avoided. This experience leads to the conclusion that in order to be efficient, didactic reversal has to occur as soon as possible. Of course, despite this error, post-test results show that the three students established an equivalence between straight lines and linear equations.

Didactic reversal was also profitable in its application to each intermediary concept. It occurred spontaneously in the introduction of ordered pairs and established their equivalence with points in the plane. Asking students to draw staircases and straight lines corresponding to given slopes certainly re-enforced their understanding of slope. The question regarding the invariance of slope helped in conceiving of this invariance as a characteristic property of straight lines. There was only one instance where the reversal did not occur. By deriving the linear equation from ordered pairs it was believed that the inverse process, generating ordered pairs from the equation, was going to be spontaneous. This was not the case and students had to be taught instrumentally how to find solutions. Even then, these were not identified immediately as ordered pairs.

The version of the teaching experiment used in the two projects proved to be extremely valuable in studying the student's thinking during his construction of new concepts. This version differs from the one used by Steffe (1979) who teaches small groups of young children who are then interviewed individually. It also differs from the one used by Rachlin (1979) who does not teach himself but interviews individually some students from a class taught by a teacher. A major advantage of the one-on-one teaching experiment is that it provides the spontaneous answers of the pupil during instruction while eliminating the many influences that group interactions can have on his thinking.

One of the problems encountered with this methodology relates to the equilibrium that has to be maintained by the interviewer between his role as a teacher and his role as an observer. Quite often, the interviewer can be so engrossed in teaching that he fails to notice some key indications of the student's thinking. This can be remedied by the constant presence of a second observer who could complement the interviewer's observations. Of course, this can create other problems such as distractions caused by his mere presence or increased shyness due to the presence of two adults. The analysis of the protocols is another reason for teaming up with other researchers. This analysis can be extremely lengthy due to the bulk of the data (600 pages of protocols,150 pages of written work on linear equations). The protocols have to be read many times until common thinking patterns can be discerned. And even then, these observations need to be tested in discussions with other investigators.

The two experiments reported here have identified some of the problems related to formal understanding. As pointed out earlier, students seemed to lose much of the formal understanding they had acquired. Of particular interest is the question regarding their difficulties in operating on equations or in generating them. Implicit in this research has been the assumption that by constructing meaning for the new algebraic forms and for algebraic operations, the learner would not experience any difficulty in using them. However there is some evidence to the contrary:

- In the project on first degree equations in one unknown, students conserved meaning for equation and could justify the algebraic operations yet could not develop on their own the necessary solution strategies. In the post-test they were unable to solve equations before being reminded by a written-out solution.

- In the second project on linear equations, students had difficulties in using their equations to generate ordered pairs although these ordered pairs had enabled them initially to derive the equations. On the post-test, the three pupils could recognize and verbalize the characteristic restriction on the coordinates of points on a horizontal line, but could not write down the equation on their own. They conserved the concept of slope and were aware of its invariance, yet they could not remember or derive the formula for slope. Although they could identify a given equation ($y = 2x$) with its graph, they could not generate any of the other linear equations.

These observations indicate that although students have constructed meaning for equations they still experience serious difficulties in operating on them or in generating them. Whether these difficulties are caused by a lack of familiarity or are inherent in the symbolic nature of algebra are still open questions.

This research has some valuable pedagogical implications. The two teaching outlines are accessible to students of varying mathematical ability. They can be adapted for classroom use by teachers who seek alternate approaches which build on their pupils' existing knowledge.

These implications are not restricted to the learning of algebra. Carolyn Kieran's observations of the students' operational thinking explains why they have so much difficulty in grasping the need for bracketing some algebraic operations. Her paper suggests how teachers can mobilise the students' sequential thinking to introduce them to a more compelling approach to bracketing and to the order of operations.

In the second project, the improvements observed during the post-test at the intuitive and relational levels of understanding are somewhat surprising. They seem to confirm Piaget and Inhelder's theory of a constructive memory which postulates that memory can actually improve with time since it may permit the learner to reconstruct certain connections not merely forgotten but unnoticed in the first place (Piaget and Inhelder,1973). In his new book on cognitive development, Flavell (1977) states that "According to this theory, the act of comprehending and encoding into memory is a Piagetian assimilation-type process of construction... Similarly, retrieval is conceptualised as an equally active and assimilative process of reconstruction, rather than as a passive, unedited copying out what is stored in memory.". If intuitive and relational understandings lead to these constructive processes, shouldn't teachers lay greater emphasis on these particular aspects of their presentations?

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Entry Behaviour:
The First False Assumption

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The work described in this paper was carried out by a team, whose members are: Rachel Israeli, Alex Friedlander, Nurit Hadas and the author (team leader).

1. Introduction

At whatever stage the student is, he is always assumed to have certain knowledge; that is.the necessary entry behaviour to what he is about to study.

Competence in basic arithmetic skills is fundamental to any mathematics program in junior high school (grades 7-9). Moreover, these skills are also necessary for the learning of other subjects as well as daily life.

In the past, under this heading of basic skills, there was a tendency to demand mastery in complicated computations. Now, as the pocket calculator becomes to be used more and more, the stress has shifted more to the understanding of the operations, developing the ability to estimate and increasing the awareness of the significance of the answers. (NIE, 1975; NCTM, NCSM, 1978)

The maths group in the Science Teaching Department in the Weizmann Institute, is responsible for the development and implementation of the maths program for junior high school. This program is close to the American classic "new maths" programs, and has been developed over the last 12 years. There are three parallel sets of material, intended for three ability levels.

Education in Israel is more or less centralised and schools choose between the available officially approved programs, of which the Rehovot program is one. More than 60% of the students in grade 7 to 9 are learning the Rehovot program.

The original syllabus for this program was based on the assumption that the student enters grade 7 with a certain body of knowledge and degree of technical competence. This assumed entry behaviour included elementary knowledge of the following topics: integers, decimals, fractions and intuitive geometry.

In reality, as everyone knows, such a global assumption is bound to be false. Many students arrive without the assumed minimal knowledge, and immediately run into trouble in their learning of the junior high school program. As usual, there are undoubtedly many and varied reasons, relative to student, teacher and school, but it seems to us that a major reason for a lack of knowledge in large sections of the population, arise from a basic learning difficulty; the first learning of those topics in primary school is insufficient for many students. In order to achieve mastery they need to learn and relearn spiral fashion.

The false assumption of certain entry behaviour and the belief that we need to provide a second chance to learn these topics, gave rise to the concept of a remedial program, specially designed to facilitate the relearning of computational skills previously studied, but not mastered.

The development and evaluation of such a program devoted to the relearning of fractions is presented in this paper. The fractions program serves as a model for the development of similar programs, which are part of the "creative implementation" philosophy of the whole mathematics group (Bruckheimer, 1978).

2. Stage I - Needs Assessment

This stage was described in detail in a report to the Ministry of Education in Israel (Israeli and Hershkowitz, 1979). Before the development of a new program, the developers have to define carefully and in detail the necessity of the proposed program. In this paper this means defining the lack of basic knowledge in computational skills within the subject of fractions. We measured this under three headings:

a) The differences between ability levels and grades 7 to 9. (It is worth examining whether there is any improvement as a by-product of learning the regular program, without remedial learning.)

b) The differences between sub-populations ("normal" and socially deprived).

c) The differences between sub-topics.

The target population of this research includes every school which takes the "Rehovot Program". The Israeli population is very heterogeneous and includes a large section which is officially defined, on the basis of socio-economic criteria, as socially deprived (s.d.). There are such s.d. students in every school; their percentage of the whole school population varying from close to zero to 100%. In order to obtain a representative sample, we arranged the schools in rank order according to the percent of s.d. students, and then took every ninth or tenth school.

The aims and objectives of the remedial program were defined according to the attitude of minimal competence described in the introduction. We began with the development of an objective achievement test which was piloted in November 1977. The resultant modified version was applied to the sample in grades 7 to 9 in all three ability streams in March-April 1978. The sample consisted of a total of 8020 students, 52% in the A-stream (the most able), 34% in the B-stream, and only 14% in the C-stream. It is interesting to note that these figures contradict a widely held belief that the Israeli streaming system tends to push most students into the middle stream: an example of the danger of unsupported assumptions, whose only strength lies in the number of times they are repeated.

We will now describe briefly the main results of the survey.

2.1. The differences between grades and levels

It is generally accepted (Lewy and Chen, 1978) that a student has reached mastery if he reaches 80% in a well-designed achievement test. We found that only 52% of the whole sample population achieved mastery.

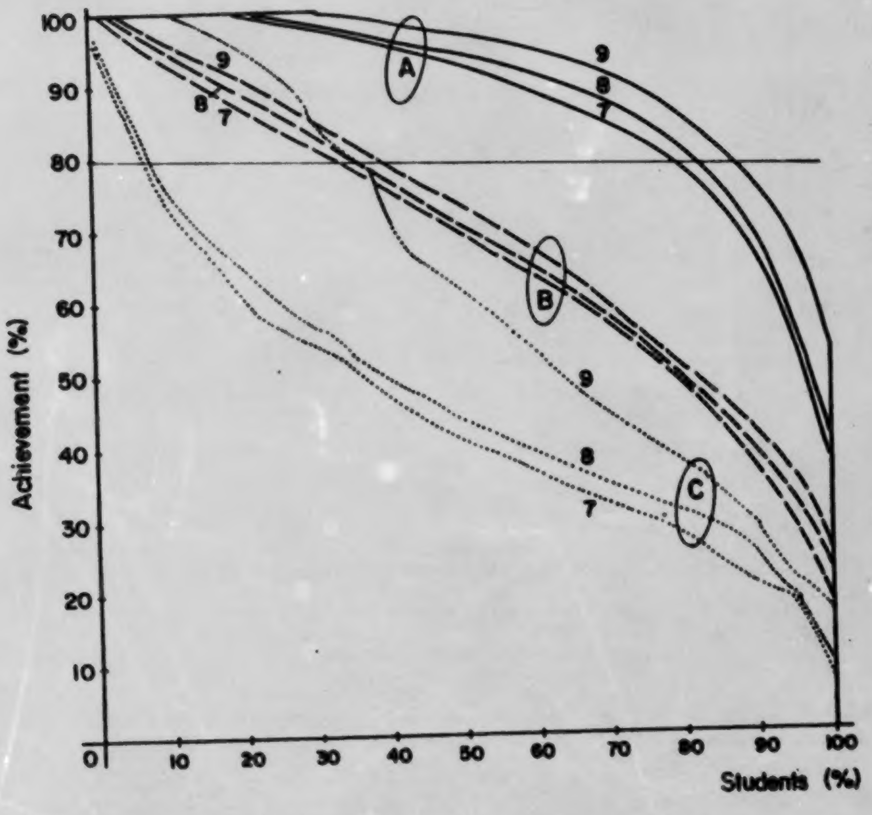

Figure 2.1.: Cummulative frequency of achievement according to grades and streams.

From Fig. 2.1. it can be seen that the variation in achievement between the three streams is large: whereas 77% of A-stream achieve mastery even in grade 7, in the B-stream only between 30% and 40% achieve mastery and in the C-stream, grades 7 and 8, as few as 10% reach mastery. The variation between grades is relatively small, indicating that competence improves very little as a result of learning the existing junior high school program, in the absence of remedial materials. Only in the C-stream there is a significant improvement between grades 8 and 9 which can be reasonably related to the systematic treatment of fractions in the existing program. This is an encouraging result, because it shows that even very weak pupils can improve their basic skills.

2.2. The differences between sub-populations

Under this heading we wished to investigate how mastery is related to the percentage of s.d. students in each school.

We divided the population into three groups according to the percent of s.d. in the school population. The results are given in the following table:

% of socially deprived student in school	% of student population achieving mastery
in excess of 70%	33%
between 35% and 70%	39%
less than 35%	66%

A different analysis of the data is given in Fig. 2.2.

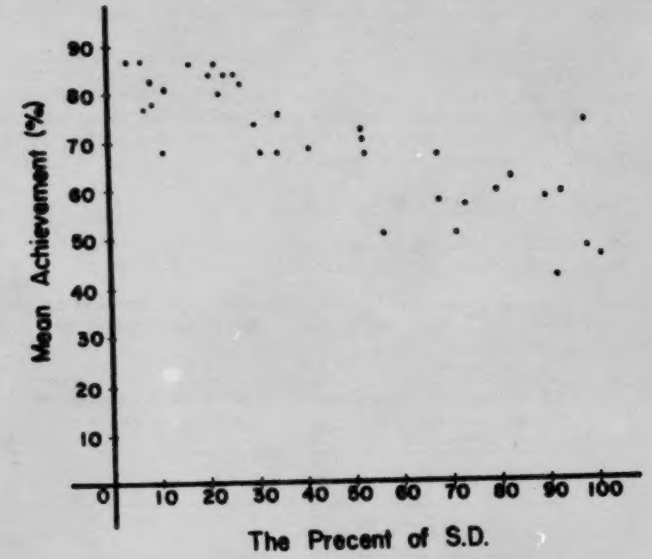

Figure 2.2.: Correlation between the mean achievement and the percent of socially deprived (s.d.) students in school.

Every dot in the graph represents one school. The correlation coefficient is -0.77, showing a clear connection between student achievement (in fractions) and the percentage of s.d.

2.3. Differences between sub-topics

The test items were divided into 4 sub-tests:

a) addition and subtraction, b) multiplication, c) division, d) the fraction concept and verbal problems. (Factor analysis on the test items validated this division.)

The mean achievement in the sub-tests are given in Fig. 2.3. Every sub-test is represented by 3 graphs, one for each ability level.

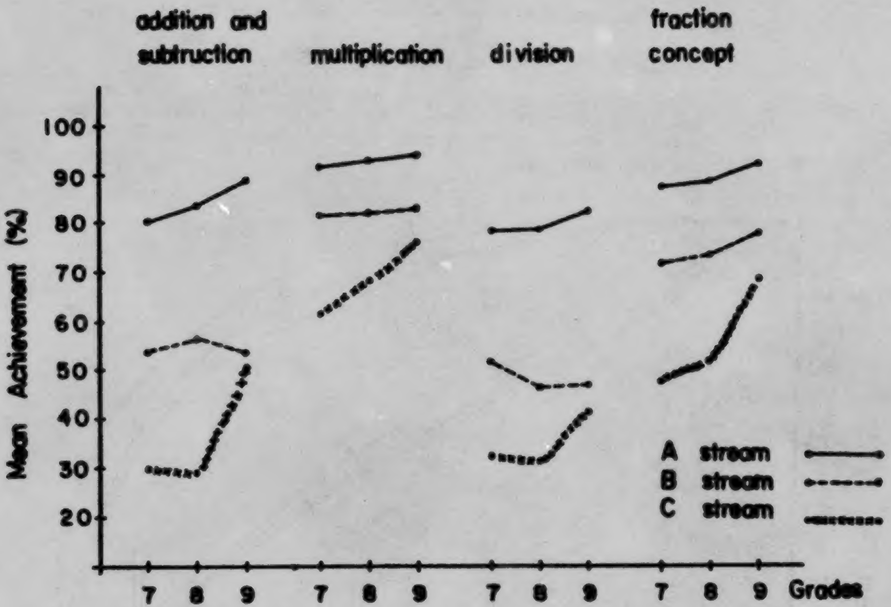

Figure 2.3.: Mean subtest scores according to grades and streams.

The achievement gap between the B/C streams and the A-stream in the multiplication and concept sub-tests narrows with the increasing grade, whereas in the other sub-tests, not only does it not narrow but it even widens a little. A more significant point is that the achievement of grade 9 B-stream is very close to that of the C-stream. This is another example of an absolute contradiction to a widely held assumption that the knowledge and potential of B-stream students is closer to that of A-stream students than those in the C-stream.

In summary we can say that there is an evident need for a remedial program in fractions, and that the need increases as the ability level decreases and the percent of s.d. students increases.

3. Design of the remedial program

The remedial program is designed to interleave with the regular, existing junior high school maths curriculum.

We had in mind the following:

a) The variation within a single class: achievement obviously varies between students.

b) The emotional factor: the student has studied the subject before and not achieved mastery. He can, therefore, be easily frustrated if asked to relearn.

c) The time factor: there is pressure on mathematics teachers to complete the existing program, therefore they cannot devote much time to remedial programs.

Each of these factors had a direct influence on the development of the remedial program.

In the first place we "perfected" a diagnostic test - accompanied by a simple table to facilitate the teacher's analysis of the results, which will determine who in his class needs remedial learning and in what sub-topics (factor a above).

Then we developed the remedial material itself, the content of which has been kept to the barest minimum and focuses on general mathematical principles such as estimation and problem solving skills within the topic of fractions. It consists of a sequence of small units, which systematically cover the subject matter. Every unit is designed to take only 10-15 minutes out of a maths lesson of 50 minutes: time which can easily be spared and probably goes to waste in most lessons anyway. Some of the units can also be used as homework (factor c above).

The teaching strategies in these units are based on individual or group learning. These strategies were developed in a special implementation project, (Hershkowitz, 1978), and were found very motivational (factor b above).

The most characteristic types of units consist of games and worksheets which contain some "prize" for example:

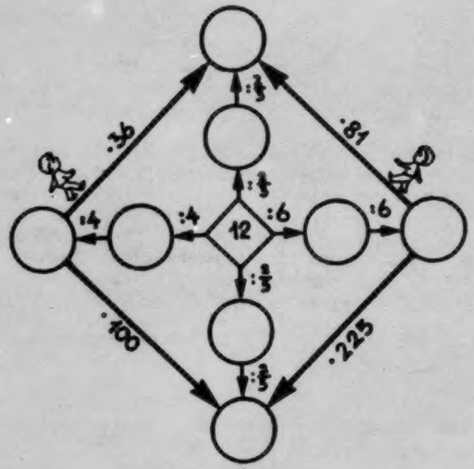

Figure 3.1.: Part of worksheet which contains mathematical "prize".

The material is flexible. It can be used with the whole class, with small groups of students or with individuals, depending on the analysis of the diagnostic tests (factor a above).

4. Development and evaluation

The sub-units were developed within the classroom in a few experimental groups located in schools with more than 70% s.d. In these classes almost every student needs remedial learning of the whole subject. We observed the teaching of the units, redesigned unsuitable units and checked them in other experimental groups. We checked the learning time of each unit, its "optimal place" in the lesson, and the optimal frequency of the units. The usefulness of the remedial program in the cognitive domain was measured by a pre-test, post-test, control group design. It was found that the pupils in the experimental group improved their mean achievement by more than one standard deviation (the mean increased from 53% to 75%), while the control group remained effectively static (the mean of the control group rose by only 3%). The achievement in the experimental group remained stable after more than half a year. The effect of the remedial program in the affective domain was checked by observation in the class and by interviewing teachers and pupils. The overriding reaction of most teachers and pupils was a sigh of relief that at last relevant material was available.

They found these ways of teaching motivational, introduced variety, made the lesson more efficient and even provided interesting challenges. It was found that the learning of the regular curriculum did not lag.

The underlying ideas of the program now serve as a model for the development of the subsequent programs: operations with integers, decimals, and basic geometrical facts. In the future we will have to check if those remedial programs maintain the promise indicated in their pilot testing. This means, we have to determine whether the teachers use the units and if so, use relevant teaching strategies and, most important of all, whether there is a significant improvement in basic skills in the overall population.

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THE EFFECTS OF RIGIDITY ON THE SOLUTION OF GEOMETRICAL "PROBLEMS TO PROVE"

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The difficulties encountered by students of synthetic (Euclidean) geometry were not investigated thoroughly, and most of the teaching modes and curricula lack theoretical justification in either learning or cognitive functioning theories (e.g. OISE, 1968; Steiner, 1971; NCTM, 1973). Most difficulties appear when students attempt to find the solution of "problems to prove", in which they are required to justify one proposition by other established ones, manipulated according to logic rules. In this situation it is not uncommon for teachers to observe either of the following behaviors: a. The student is iteratively applying previously learned method which is improper for the current problem (or subproblem) and does not change it although he is fully aware of its impropriety. b. The student applies a proper previously learned method, or a combination of such methods, which are very lengthy, in spite of the existence of an immediate shorter method.

The main psychological approaches (i.e. psychoanalysis, field theory of K. Lewin, Gestalt theory, associative theories) dealt with such behaviors which they named rigid. Researchers within these theoretical frameworks demonstrated that in certain conditions rigidity impairs problem solving performance on insight problems and anagrams. These kinds of problems differ substantially from geometrical "problems to prove".

Geometrical "problems to prove" are usually stated in search representation, hence they are solved by heuristic search (Newell & Simon, 1972). Solution methods are of utmost importance in solving problems (e.g. Reitman, 1965; Newell & Simon, 1972), but their exact role in solving "problems to prove" was not yet investigated. The failure to find a proper method to be applied to a node (current state, or knowledge state) in the search tree may stem from either of the following sources: a. Reasoning errors committed either in the solution phase (while judging the suitability of a method) or in the evaluation phase (in checking the correctness of the solution path). b. Use of an incorrect meta-plan, strategy or heuristic for deciding upon the method to be applied.

It is the aim of this study to propose a model to account for the second source of difficulties. This model is a development of Duncker's (1945) functional fixedness (FF) and Scheerer and Hulling's (1960) cognitive embeddedness. Duncker's FF proved as an adequate construct for explaining why certain perceptual characteristics of problems inhibited some thought processes (e.g. Duncker, 1945; Glucksberg & Weisberg, 1966), and we can show that FF may also operate in the solution of geometrical problems. For instance, consider a problem in which the input for a method contains a side of a triangle, and the candidate line segment serves also as a height in another triangle. It may well happen that the student perceives the line segment, but cannot use its "side" function which is fixated by its "height" function. Hence, the method cannot be applied, since the line segment cannot be used as input. But FF cannot explain why a student fails to chose, apply, adopt or accomodate solution methods that have no perceptual components, or why he iteratively applies improper methods.

It was shown experimentally by Scheerer and Hulling (1960) that FF cannot account for the fixation of thought material, so it was necessary to propose a new rigidity construct. Their construct, named cognitive embeddedness (CE), refers to the inaccessibility for the subject of the meaning of an object (idea, notion, concept). This meaning lost its independance because it was included in a meaning unit with other object(s). Scheerer and Hulling suggest that the meaning unit has to be broken into its components in order that these components' meanings will regain their independance. Only then they become accessible to the subject for use in other contexts. According to Scheerer and Hulling, meaning units can be formed consciously or unconsciously, and temporal or spatial proximity of the objects will suffice for its formation.

We hypothesize that meaning units of a different kind produce CE that interferes with intellectual processes in solving geometrical "problems to prove". Such meaning units are groupings of one (or more) solution method(s) with one (or more) knowledge state(s). Like meaning units groupings are formed as the consequence of the subject's actual (or even considered) application of a method to a certain knowledge state, regardless of the appropriateness of this method or the consequences of this application. We shall call such a unit "coupling". A method which is coupled with a knowledge state is applied to this state in exactly the same way as it was applied (or considered to be applied) to it in the original situation, in which the coupling was created. In order to apply a method to a new knowledge state, the method has to be released from a coupling which includes it, and then it can be applied to the new knowledge state. The decision as to how to apply this method is based on the strategies and heuristics the student has, and the decision which method is proper, in a current state, is based on the perceptual, semantic and/or syntactic cues in the problem statement and/or representation. After the method to be applied was chosen, the student searches his memory, in which these couplings are stored, after couplings that contain this method. If the coupling(s) he has found actually contain both the current state and the chosen method, then this method can be immediately applied to the current state. But if the coupling(s) he has found contain the chosen method and differnt state(s) than the current one, then the method cannot be applied directly to the current state, unless it is released from at least one of these couplings. The release can be achieved through reflective experience with this coupling. If this effort was rendered successful, then the student can apply the released method to the current state, thus transforming it into another one, making a progress in the search tree.

The preceding analysis distinguishes between two geometrical rigidity constructs. Such distinction is both justified and càlled for by Levitt & Zelen's (1953) arguments for indexed rigidities. We named the perceptual rigidity as Geometrical Functional Fixedness (GFF), and the intellectual one as Geometrical Method Rigidity (GMR). The first is Duncker's well established functional fixedness operating in geometrical contexts, and the second is the fixation of one method by the coupling in which it is contained. The result of GMR is inability to use an appropriate method instead of the inappropriate applied method, or inability to use shorter method than the applied one. Geometrical Rigidity (GR) is the construct comprising both GFF and GMR.

The definition of GMR strongly resembles those given in rigidity literature (e.g., Adamson, 1952, Cattell, 1935, Duncan, 1959, Fisher, 1950, Goodstein, 1951, Guetzkow, 1951, Luchins, 1942, Rokeach, 1948, Wesley, 1953), but it is, nevertheless, different from them in important aspects. *a*. The term "method" was used in these definitions in a rather vague sense, in contrast to the exact meaning it has in information processing theory (e.g., Reitman, 1965, Newell & Simon, 1972). *b*. Rigidity was thought of, and treated as a general trait, but this was done without theoretical and empirical justification (Leach, 1967), thus producing numerous contradictory empirical results (Hoz, 1976). GR is restricted in our definition to certain intellectual and perceptual aspects of geometrical problem solving processes.

In order to explore and estimate both the common and the different aspects between GR and other rigidity constructs we employed three rigidity measures, each representing one of the main psychological approaches: Gestalt theory (Breskin Rigidity Test, Breskin, 1968); Field theory of K. Lewin (Self Description Inventory, Braen, 1960) and cognitive syle approach (EFT, Witkin et al., 1962). These measures seem to be relevant to the processes in the solution of geometrical "problems to prove", since these processes involve verbal and non-verbal, perceptual and intellectual processes. BRT and EFT are non-verbal perceptual tests, but the first is a preference test, while the other is an ability test. SDI is a typical personality test because it is verbal subjective self-description that involves no reasoning processes. EFT was used as a rigidity measure by Gardner et al. (1959), because it was found to be contained in a rigidity factor. We employ it in order to check this finding, to find correlates to it and to test a hypothesis raised by Duncker (1945), namely, that the weaker the student is in mathematics, the lower will be his score in an EFT like test (Gotschaldt Test), because his concepts are saturated with perceptual images. Breskin and Gorman (1969) found that field dependent males were more rigid on BRT than field independent males, and the opposite result was found for females. Cattell & Tiner (1949), Zaks (1954) and Fenchel (1958) report positive correlations between water jar rigidity and field dependence. Busse (1968) found that EFT belonged to a general rigidity factor, that differed from spontaneous flexibility. On this basis, four sets of hypotheses were tested. *Construct validation of rigidity*: Negative correlations were expected between BRT and EFT and between SDI and EFT, and a positive one between SDI and BRT. *Relation of rigidity to geometry learning*: Positive correlations were expected between EFT and school geometry scores, and negative correlations between school geometry scores and both BRT and SDI. *Construct validation of GR*: Negative correlations were expected between GR (and its components) and EFT, and positive correlations between GR and both SDI and BRT. *Relation of GR to geometry learning*: Negative correlations were expected between GR (and its components) and school geometry scores.

METHOD

Subjects: 48 ninth grade students, comprising two classes of high school in Jerusalem.

Procedure: Six tests were administered (in mathematics classrooms) to each class: GFF, GMR, geometrical abilities, EFT, BRT and SDI. GR was printed in two parallel forms (which proved to be equivalent on later analysis). The reactive arrangements of the research were minimized through teachers' administration of geometrical abilities and GR tests as regular mathematics tests. Geometrical abilities test was administered two months before the end of the year, GR tests - one month later, and the rigidity tests and EFT - on the last mathematics classroom of the year. Geometry scores for each student were recordered from his student file.

Variables:
1. *Geometrical abilities*. After Bloom (1956), a test of five geometrical abilities was constructed: Knowledge, Comprehension, Analysis, Synthesis and Evaluation. The total test score was the sum of the first two and the last two subtests (The reason will be explained later). The maximal score was 60.

2. *GR*. GR test was composed of two subtests: GFF test and GMR test. GFF test contained items depicting complex geometrical drawing, requiring the student to find and/or identify examples of the concepts "exterior angle" and "height of a triangle". A correct response got a score of 1 and an incorrect one - 0. GFF score was computed as a difference between the maximal score and the sum of scores the student got, so that higher score reflects higher degree of GFF. Items of GMR test were

simple "problems to prove" that can be solved without construction (auxiliary lines). The features of these problems that enable the measurement of GMR are: a. Each can be decomposed into at least two subproblems. b. Each has at least two different solutions. c. One of the solutions is direct (shorter) and the other is indirect (longer). d. The direct solution is a "non-standard" solution[1], meaning that the student does not have established couplings that contain the direct method and the current state, so that the direct method should be released from established couplings. The indirect solution is a "standard" solution, meaning that it can be attained by successive applications of methods that either (i) were included in established couplings to the knowledge states that are produced by the solution of each subproblem, or (ii) were coupled with knowledge states that were very similar to that of the test item, so that the required transfer of learning is minimal. e. The problem statement, especially the diagram, is very suggestive for attempting the indirect method, because it resembles problems with which the student was familiar. It was conceived that the degree of GMR is inversely proportional to the number of solved subproblems, since GMR can operate at each node of the search tree, and the student should overcome its operation at every node. Hence, the longer the advance in the search tree of the indirect solution, the lower the degree of GMR. The direct solution indicates no operation of GMR. The scoring was as follows: For the direct solution the student got 0 score, and for indirect solutions, the score was computed as the difference between maximal score and the sum of scores on all subproblems. (For each solved subproblem the student got a score of 1.) Thus, higher score reflects higher degree of GMR. The maximal scores on GR were 21 (Form A) and 29 (Form B).

3. Rigidity measures. The various sorts of rigidity were measured by GEFT, BRT and SDI. The maximal scores were 15, 15 and 49, respectively. Higher EFT score reflects higher degree of field-independence (and lower degree of field-dependence), and higher scores on SDI and BRT reflect greater rigidity.

4. School geometry scores. These are three trimester scores, assigned by the regular geometry teachers during the year. The maximal score for each trimester was 10.

RESULTS

First we checked that GR did not affect the performance on geometrical abilities test. That was done since the analysis subtest of geometrical abilities test contained items similar to those in GFF test, and since GMR and GFF were assumed to operate in the solution of geometrical "problems to prove". It was found, by multiple regression analysis, that the contribution of GFF to geometrical abilities scores was not significantly different from zero, and that GMR contributed 9.29% ($p<.05$) to the above variance.

Table 1 shows the intercorrelations between three sets of variables: Rigidity measures, geometrical rigidity and school geometry trimester scores. Regarding the construct validation of rigidity, BRT and SDI were positively correlated ($r = .31$, $p<.05$), thus confirming our hypothesis. Contrary to our hypothesis, EFT was found to be uncorrelated with both BRT and SDI.

Hence, BRT and SDI seem to measure different "rigidities", and EFT seems not to be a rigidity measure.

EFT was found to be positively correlated with two school geometry trimester scores, as hypothesized, but contrary to our expectation, uncorrelated with trimester B scores.

Table 1: Correlations between rigidity measures and the other variables.

	rigidity			geometrical rigidity			school geometry		
	SDI	EFT	BRT	GFF	GMR	GR	Trim.A	Trim.B	Trim.C
SDI	-			-.30*	-.34*	-.34*	.34*	.30*	.36*
EFT	.23	-		-.50**	-.46**	-.54*	.42*	.28	.48**
BRT	.31*	.20	-	.19	-.14	.01	.12	.04	.18

Note: * $p<.05$, ** $p<.01$

Two additional findings are in contrast to our hyptheses. SDI was found to be positively and BRT - negatively correlated with school geometry scores.

Table 2 depicts the portions of school geometry scores variances accounted for by BRT, SDI and EFT.

Table 2: Contributions of various rigidity measures and field independence to the variances of geometrical rigidity and school geometry.

contributing variable		criteria					
		Trim.A	Trim.B	Trim.C	GFF	GMR	GR
SDI	a)[1]	11.42	8.81	13.02	8.83	11.52	13.23
	b)	11.68	8.87	13.06	8.88	9.58	13.30
	c)	10.08	8.93	6.62	8.68	10.19	9.36
EFT	a)	17.40	-[4]	23.26	24.54	21.03	29.13
	b)	14.43	-	21.20	20.58	18.02	24.71
	c)	12.31	-	16.31	19.38	12.71	20.06
GFF	a)	14.69	20.43	22.16	-	-	-
	b)	10.25	14.36	19.30	-	-	-
	c)	9.94	9.58	6.12	-	-	-
GMR	a)	15.81	16.18	24.30	-	-	-
	b)	11.08	11.40	21.60	-	-	-
	c)	8.38	16.03	6.05	-	-	-
GR	a)	19.16	23.28	30.08	-	-	-
	b)	14.13	16.37	27.11	-	-	-
	c)	11.77	10.73	8.58	-	-	-

Notes:

1) All the contributions are significant beyond .05 level.
2) The net contributions of the current variable, beyond its common variance with geometrical abilities.
3) Net contributions of the current variable, beyond its common variance with rigidity measures (SDI, BRT, EFT), disregarding the magnitude of separate contributions.
4) - means: contribution of 0%.

The main contributions of SDI and EFT, including the common variances to all other rigidity measures, are about 11% and 13% respectively. But the net contributions are about 8.5% and 9% respectively. BRT contribution is zero.

As expected, EFT was found to be negatively correlated with GR and both its components, and to account for about 25% of their variances including its common variances with BRT and SDI, which was estimated to be about 4%.

SDI accounted for about 9% in GFF variance and for about 10% of GMR variance. These percentages did not change after the removal of the common variances it shared with EFT and BRT. With GR variance the decrease, owing to the extraction of these common variances, was about 4%, indicating a small overlap between GR and EFT and the other rigidity measures.

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[1] Which solution is "standard" and which is "non-standard" was decided upon by examination of problems in textbooks used in these classes.

As expected, GR constituted a serious hindrance to geometry learning, as is evidenced by the negative correlations in Table 3.

Table 3. Correlations between geometrical rigidity and school geometry achievement.

	GFF	GMR	GR
Trim. A	-.38	-.39	-.44
Trim. B	-.45	-.40	-.48
Trim. C	-.47	-.49	-.55

Note: All correlations are significant beyond .01 level.

It can be seen in Table 2 that about 4% of the variance contributed by GR to the variances of school geometry are common to GR and to geometrical abilities scores.

On the other hand, the portions of school geometry scores variance accounted by GR is seriously decreased in 1% to 19% by the extraction of the contribution of SDI, BRT and EFT. The rigidity measures, namely BRT and SDI, explain about half the contribution of EFT in trimester A scores variance and about the same percent in trimester C scores variance. These establish the independent existence of GR, whose portions in the explained school geometry scores variances are about the same as those of field dependence-independence cognitive style.

Another means for the establishment of GR was through stepwise discrimination analysis, the results of which was depicted by Table 4.

Table 4. Means and S.D. of most rigid group and moderate and lowest rigid group, on the two forms of CR test scores.

Variable^1	Form A					
	most rigid(10)		others(13)		F^2	F^3
	M	S.D.	M	S.D.		
Trim. C	5.80	.79	7.00	.91	10.96**	10.96**
EFT	6.20	3.19	11.23	4.23	6.07*	9.84**
BRT	9.10	2.69	7.08	3.12	1.07	6.94**
Trim. A	5.90	.99	6.77	.93	1.53	5.73**
Geometrical abilities	41.30	23.19	53.46	24.68	.25	4.45**
Trim. B	5.70	1.16	7.15	1.14	.02	3.50**
SDI^4	26.20	4.71	29.08	5.71	-	-

Variable^1	Form B					
	most rigid(9)		others(12)		F^2	F^3
	M	S.D.	M	S.D.		
Trim. C	5.33	1.12	6.81	.91	12.92**	12.92**
EFT	7.44	4.22	10.31	4.36	4.38*	9.60**
BRT	9.89	3.92	8.94	3.82	2.28	7.53**
Trim. B	5.87	1.09	6.88	.96	2.20	6.52**
Geometrical abilities	48.41	19.38	55.44	15.52	1.07	5.45**
SDI	18.56	10.83	27.63	5.44	.63	4.56**
Trim. A	6.00	1.12	6.56	1.09	.73	3.95**

Notes: * $p<.05$, ** $p<.01$

1) Variables appear in their order of entrance into discrimiation equation.
2) F ratio for ANACOVA in which current variable is the criterion and the preceding variables are covariates. This ratio reflects the discrimination power of the current variable alone.
3) F ratio for multiple ANOVA on all variables including the current one.
4) SDI did not reach the tolerance level and was deleted from the analysis.

For each form of GR test we created two groups of students. a. the most rigid on GR, b. all the others (moderate and lowest on GR). These two groups could be separated by disciminant functions, in both forms, on almost all the variables. But EFT and trimester C scores could by themselves discriminate the two groups. The profiles of the two groups of students show that the most rigid (on GR) ones (1) score lower on school geometry achievement and geometrical abilities test, (2) are slightly more rigid on BRT, (3) have higher field dependence scores and (4) are rigid on SDI as the other students. This established the GR construct as both distinct from, and still having commonality with SDI, BRT and EFT.

DISCUSSION

The two proposed geometrical rigidity constructs, GFF and GMR, proved to be independent of each other, thus reflecting the separation between perceptual rigidity and intellectual rigidity in solving geometrical "problems to prove". These constructs overlap SDI rigidity slightly and articulated-global cognitive style - considerably. They constitute a hindrance to school geometry learning and are good predictors for future failure in school geometry studies.

It is not surprising that GFF scores and EFT scores were closely related, since they pose the same demands on the subject. This is an affirmation of Duncker's (1945) prediction that the weaker the mathematician, the lower his performance on Gotschaldt Figure Test. He hypothesized that the concepts of the weak mathematician are saturated with perceptual images, that prevent him from identifying and using these concepts in an embedding context. This hypothesis found support in the mean correlation between EFT and Mathematics subtest of SAT: .39 ($p<.05$) (Witkin at al. 1977). Nevertheless, GFF performance differs from EFT performance in that the figures used in GFF test are examples of established geometrical concepts, while for those in EFT the subject has to create new inner representations, and store them in his memory.

The close relationship between the performance in EFT and GMR test confirms our assumption about the formation, breakdown, utilization and storage of couplings. We further assume that the complexes of couplings stored in LTM and STM are analogous to the diagrams of EFT. In order to perform well on GMR test the subject must articulate his memory in order to find suitable couplings, and he must be able to break such couplings into their components. This is also the same articulation he has to do in EFT.

It can be argued that performance on GR tests may well depend on perserverance, on ordered habits and other like behaviors that are measured by SDI. This expectation was confirmed by the positive correlations between SDI and school geometry scores. For these reasons it was quite surprising to observe negative correlations between GR and SDI. The reason for these conflicting results may be that while school geometry scores are confounded with such factors, GR test scores are not, and constitute a more objective measure of geometrical performance. With BRT the theoretical confusion is even greater, since it is based on the assumption that the more rigid the subject is, the greater structure he seeks and imposes on his environment (Breskin, 1968). According to Witkin et al. (1977), the more field independent the subject is, the greater is his ability to articulate his environment. Thus, a positive correlation has to be predicted between EFT and BRT. Our findings are in favour of this hypothesis only for males ($r = .49$, $p<.01$) but not for females ($r = 0$), just the contrary to Breskin and Gorman's (1969) results. These conflicting results may be due to the nature of the two tasks: EFT is an ability test, geometry achievement is, in part, dependent on geometrical abilities and information processing capacities, whereas BRT is a performance test. The empirical evidence in favor of the construct validity of EFT and the low correlation between SDI and BRT, that indicates weak construct validity of rigidity, seems to invalidate BRT as a rigidity measure, as well as a psychometrically.

It has often been claimed (e.g. Maier & Burke, 1966, Bourne et al., 1971) that personality and cognitive sytle variables are uncorrelated with problem solving abilities. On the other hand, the comprehensive survey of Witkin et al. (1977) shows that field dependent subjects achieve less on insight problems than do field independent subjects. The results of this study are in accord with the latter finding.

Research on rigidity showed that in certain conditions rigidity does impair the ability to solve several sorts of problems (Hoz, 1976). Since SDI is considered as a rigidity measure, it was surprising to find that the more rigid the subject is, the higher is his score on school geometry achievement. Prof. S Armitzur[1] remarked to me that adhering during long periods of time to certain solution methods or lines of reasoning, in spite of repeated failures, is a frequent phenomenon among eminent mathematicians. These mathematicians would, no doubt, be classified as most rigid. As we mentioned earlier, this perseverence may be measured by teachers scores, as well as by SDI.

This is in line with the positive correlation of .49 (p<.01) between SDI and Lorge-Thorndike intelligence test, found by Braen & Wallen (1960) for high school students. Since BRT and SDI are only slightly correlated, it can be concluded that SDI is a measure of different kind of rigidity than is BRT. The reason why BRT was found to be unrelated to school geometry achievement may be that BRT performance is based mainly on perceptual nonverbal processes reflecting preference, whereas geometry learning is based mainly on verbal and other symbolic processes reflecting logical and problem solving abilities. This may also explain the negative correlations between GR and both SDI and BRT.

The relationships that were found between GMR and SDI and EFT suggest that GMR is not a general personality trait, but a cognitive style construct, which developed from the interaction of the subject with his environment, and it is susceptible to change with changes in the environment. We tested this hypothesis in two ways. The experimental one, which was reported elsewhere (Hoz, 1976), consisted of proposing a heuristic prescriptive model for finding and constructing proofs in geometry. This model was taught to several groups of students and produced a considerable reduction of GMR. The present statistical one was to examine the performance on similar and dissimilar items of GMR test. Similar problems either had the same diagram or the givens of one problem were the conclusion of the other and vice versa. It turned out that the correlations between two couples of similar items were .43 (p<.05) and .56 (p<.01) The correlation between unlike items were .42(p<.05), .51 (p<.01) and .30 (N.S.). Had GR been a personality trait - a much higher correlation would be expected between the similar items. The exchange of givens and conclusions produced a change in performance so it is an evidence of the importance of environmental variables and their interaction with geometrical problem solving processes.

It is interesting to note the increasing contribution of GR to school geometry achievement variance from the first to the last trimester. This phenomenon resembles the differentiation of factors in learning that was found by Fredrickson (1967). This differentiation was said to reflect changes in the strategies used by the subject in the different phases of learning. In the first trimester only few proofs were encountered by the students, so that the number of methods used was very small, and only few couplings were formed. The number of solution methods that were used considerably increased during the second trimester, resulting in a large number of stored couplings. This process was further intensified during the last trimester. Thus, the degree of GMR could increase from trimester A to C, and could be reflected in greater variance of GMR, thus increasing its weight in determining problem solving behavior. This analysis was supported by the results of the discriminant analyses, and by similar results that were obtained in a study involving another heuristic prescriptive model for geometrical constructions (Hoz, 1979).

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TEACHER ATTITUDES AND STUDENT PROBLEM SOLVING ACHIEVEMENT AFTER ONE YEAR USE OF HAND HELD CALCULATORS IN THE ELEMENTARY SCHOOL

Lucreda A. Hutton

The hand held calculator has been the center of much recent controversy among mathematics educators. In 1972 the American public purchased nearly two million electronic hand held calculators (according to an article in June 1973 "Consumer Reports.") Since that time the quality of the calculators has improved, the price has decreased, and additional millions have been sold. Today four function machines with memory can be purchased in U.S. supermarkets for under $10.00. The readily accessable calculator is a fact of life; the question is how to effectively include it in the mathematics curriculum and how the classroom teachers will respond.

The effects of the calculator on mathematics achievement and attitude are still unknown. The early studies have been primarily at the junior high (age 13-15) and high school (age 16-18) level and with low achieving subjects (Cech, 1970; Gaslin, 1972; Ladd, 1973; Mastbaum, 1969). For the most part these studies record no significant differences in attitude and achievement for subjects who had used calculators in contrast to those who had not. One exception was a group using the calculator with special text materials (Gaslin, 1972). These subjects did perform significantly better with respect to retention of skills and rate of mastery when compared with subjects who did not use the calculators. Experimentation at the elementary level is still essentially unreported. Yet the development of good attitudes toward mathematics at this level is crucial (Callahan, 1971).

As reported by Aiken (JRME 1976), when attitude scores are used as predictors of achievement in mathematics, a low but significant positive correlation is usually reported (Neale, 1969). Acknowledging the importance of student attitudes and the influence of the elementary teacher, this report will consider elementary teacher attitudes toward the hand held calculator and the achievement of their students.

During the 1976-1977 school year the author and Glen D. Vannatta co-directed a project in the Indianapolis Public Schools which involved the use of the hand held calculator in the curriculum of thirty-eight intermediate level classrooms. All teachers of fourth, fifth, and sixth grades (age 9-11) in six inner city schools participated in the project. The teachers were not volunteers and at the beginning of the year their attitudes toward the calculator ranged from extreme negativism to complete acceptance. Approximately 1000 students were in these classes.

Specifications for the calculators were written by the supervisor of mathematics. The school system accepted bids and subsequently selected the Rockwell Model 24RD II calculator. This is a basic four function machine with memory, percent, square root, and sign change keys. The keys are large and color coded. Teaching materials were supplied and are described below. Likert-type questionnaires were prepared for teacher reactions to the calculator in their classrooms. In an effort to elicit honest responses the questionnaires were not channeled through school administration but were given directly to the investigator. A summary of the responses after a trial period of one year follows.

RESPONSES TO TEACHER QUESTIONNAIRE

	% AGREE	% NO OPINION (NO CHANGE)	% DISAGREE
Classroom Management			
a) Management required very little extra effort	74	0	26
b) Security was not a problem	71	15	14
Curriculum			
a) Topics covered limited calculator use	36	9	55
b) "Problem of Week" was a valuable resource	51	37	12
c) Teachers textbook guide was helpful	50	42	9
Student			
a) Use of calculators resulted in greater interest and effort by most students	74	6	20
b) Discipline matters seemed improved when calculators were being used	41	38	21
c) The pupils like to use the calculators	91	0	9
d) Calculators improved the students' interest in word problems and problem solving	65	0	35
e) The students in general were "careful" with the calculators	94	3	3
Parents			
a) There was no evidence of parental opposition to calculator use	100	0	0
Teacher Attitude			
a) Calculators should be used in the elementary school mathematics curriculum			
1) At the beginning of the school year	50	21	29
2) At the end of the school year	76	12	12

Classroom Management. As teacher responses indicate, there was no problem with classroom management and security. Each calculator was marked with a code which indicated the school, teacher, and student responsible for the machine. The teachers used varying routines for checking in and out the calculators. However, using student monitors appeared to be the most popular and efficient method. Several teachers remarked that they were amazed at the ease of management. No calculators were damaged to the extent that they no longer functioned.

Curriculum Materials. Extemporaneous comments from the teachers were encouraged. They indicated without exception that they wanted strong curriculum support for the calculator. Teachers also wanted in-service training and in-class demonstrations to help acquaint them and their students with the use of the calculators. A pamphlet explaining how to use the hand held calculator was prepared and distributed to the teachers and students. Three types of regular curriculum support were provided for classroom use. These materials consisted of a daily textbook supplement, a weekly problem solving lesson, and games and activities. The daily textbook supplement followed the textbook lesson by lesson and contained suggestions for the limitation, as well as suggestions for the incorporation, of the calculator. Supplementary problems written especially for calculator use were supplied where appropriate. The use of calculators allowed more realistic problems since the students could manipulate large numbers with ease. For example, a word problem involving the purchase of clothing could use realistic prices which made the problems more meaningful.

The calculator thus allowed the students to concentrate on the problem solving aspect of the problem instead of being pre-occupied with the arithmetic. The teachers were encouraged to allow the students to use the calculators for all word problem lessons. The second type of support was the weekly problem solving lesson entitled "Problem of the Week". It consisted of an information sheet and an overhead transparency by which the teacher presented the problem to the students. These problems were designed to teach consumer problem solving and an attempt was made to make them relevant to the students' environment and interests, e.g. sports averages, shopping, earning money, auto racing speeds, etc. These problems gave rise to lively discussions which could lead to extensions of the problem.

These two approaches to curriculum support, textbook supplement and problem solving lessons, shared about equal popularity with the teachers. Sixty-five percent of the teachers felt that the calculators improved their students interest in word problems and problem solving. The Metropolitan Achievement Test (a standardized test) was given at the beginning and again at the end of the school year. The sixth grade students (age 12) who used calculators during the school year scored significantly higher on the problem solving subtest when compared with the sixth grade students who had not used calculators. Neither group of students used the calculators for the test.

The third type of curriculum support consisted of games and varied activities called "Calculator Calisthenics." The objectives of these activities varied from reinforcing the multiplication facts to understanding concepts. For example, a competitive game was developed where pupil A would enter a multiplication fact such as 6 x 9. Pupil B would have to say aloud the product and then push the = key to reveal the correct answer. A point was scored for pupil B if he was correct, or a point for pupil A if not. Pupils alternated in giving problems until a goal score was reached. This game took advantage of the immediate feedback of the correct answer.

Student Attitude. The teacher responses indicate that the majority of teachers believed the calculators were a motivational asset. Some teachers complained that they had difficulty ending the mathematics lessons. One complained that the students got "too excited" and "wanted to work ahead." The development of a positive attitude toward the subject being studied is probably one of today's most prevalent educational goals (Michaels and Forsyth 1978). Since mathematics has a reputation for being unpopular (Wilson 1961) the possibility that the calculator can effect attitudes deserves serious consideration. It is of special interest that teachers believed the calculators not only improved student attitudes towards the mathematics lessons in general, but specifically improved student interest in word problems.

Parent Responses. Many teachers reject the use of the calculator on the grounds that the parents would object. It is interesting that with approximately one thousand students using calculators not one teacher reported one case of a parent objecting. Several teachers reported positive comments from parents. One parent-teacher organization purchased extra batteries for the calculators at their school.

Teacher Attitude. The change in teacher attitude after one school year of calculator use is of special interest. A total of seventy-six percent of the teachers felt that the calculator should be used in the elementary school mathematics curriculum. The increase from fifty percent at the beginning of the year would correlate with the increase in attitudes found by Singleton (1972) after an in-service training program. The commonly expressed fear that the calculator eliminates the need to "think" vanishes quickly when one attempts to use the calculator. Actually the calculator can free the student from tedious computation so that he can "think" about the problem.

Ninety-one percent indicated that they felt the calculator was a benefit in their classroom as motivation, immediate feedback, or for practical knowledge. Only nine percent indicated that they believed it to be of no value in the classroom. During the supervision of this project the investigator became aware that teacher attitude seemed to be playing an important part in the amount of actual usage time occurring in the classrooms. Therefore, an analysis of the achievement data was made as follows. Pairs of teachers were identified within the same grade level and the same school such that one teacher was pro-calculator and one teacher was neutral or opposed to calculator use. There were three such pairs identified at the fourth grade level and three at the sixth grade level. All of these groups had calculators in their classrooms. The table below outlines the results of this analysis. In every case the group with pro-calculator teachers recorded higher mean scores.

COMPARISON OF DIFFERENCE SCORE MEANS FOR ACHIEVEMENT ON TOTAL MATHEMATICS SECTION OF METROPOLITAN ACHIEVEMENT TEST

Fourth Grade (age 10)	Sample	Mean	Standard Deviation	t value	Significance
Pro Teacher	55	21.24	11.58	2.318	.05 Significant
Opposed Teacher	50	15.92	11.91		
Sixth Grade (age 12)					
Pro Teacher	51	17.45	12.64	2.456	.02 Significant
Opposed Teacher	73	11.92	12.13		

Summary. The teacher factor remains the most important aspect of effective instruction. The teachers need and want curriculum support for the calculator if their attitude toward using the calculator is going to be constructive and positive. It is the author's opinion that the strength or weakness of the calculator as a teaching aid lies in the curriculum support and teacher preparation supplied, particularly in both number facts practice and the range of meaningful problems that the child can attempt. There is no question that the students "like" to use the calculators. If a favorable teacher attitude can be developed, and proper curriculum materials supplied, it may well be that the calculator can help improve problem solving skills and student attitude toward word problems in the eleven to twelve year old students. There is definite need for continued research in these areas.

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The use of situations for the development of mathematical concepts

Claude Janvier

Over the centuries, schools have always emphasised the learning of rules, laws, principles, and hoped for their "understanding" to emerge from a series of examples and exercises. On the other hand, the apprenticeship paradigm stressing the gradual construction of rules and progressive discovery of principles has also guided the activities of schools, but for many reasons only marginally. The discovery learning movement (no longer in its youth) has, however, contributed to its growing use in many curricula today. As a consequence, a new emphasis has been put on the notion of appropriate or relevant environment or on the idea of a rich situation likely to bring about discoveries or to encapsulate rich abstract ideas, simultaneously fading out the search for the most efficient and orderly theoretical presentation of topics.

In a recent research project (JANVIER, (1978)), in which we dealt with the interpretation of cartesian graphs representing situations*, we came across several questions related to the use of situations. In this paper, we shall make use of one of the tasks that we devised to illustrate practically various issues involved in the use of situations.

As a matter of fact, the conclusions to which we shall arrive could be derived from a theoretical analysis. We have (always) preferred a concrete approach based on simple examples, leaving the theoretical argumentation to the readers.

We interviewed 20 pupils with the RACING CAR problem: 7 in 1st year** (age 11 - 12), 7 in 3rd year and 6 in 4th year. The problem comprises three parts. In addition to the interviews, a written version***, involving only parts A and C was administered to about 39 pupils of first year.

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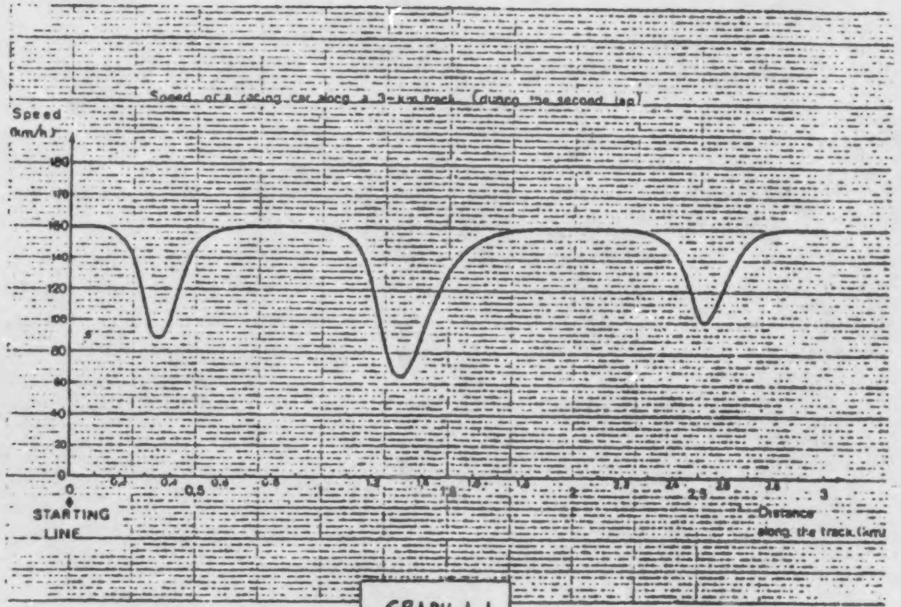
* and which will be reported in various papers

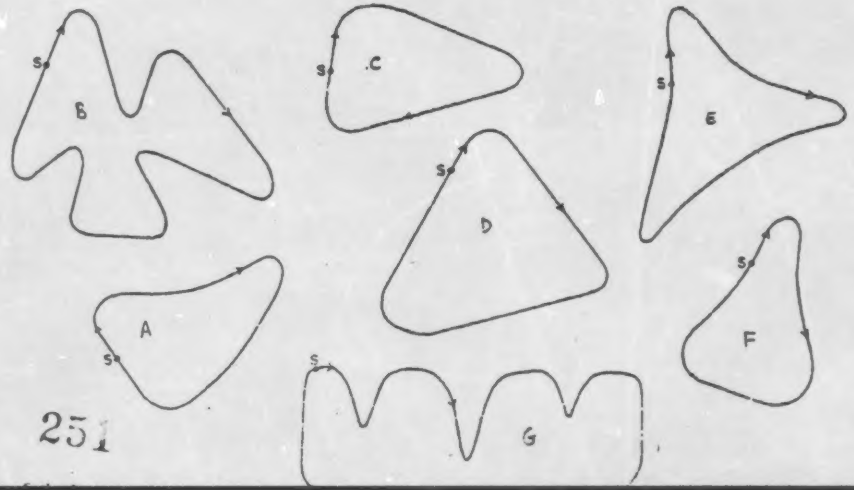
** Secondary level

*** to be handed out to the participants



GRAPH 1.1

Selection of tracks



ABRIDGED DESCRIPTION OF THE TASK

RACING CAR, Part A

Presentation (on graph 1.1)

> "As the heading shows, this graph tells how the speed of a racing car varies along a 3km track during its second lap. The speed is given in km/h. This axis (pointed along with the forefinger) gives you the distance along the track from the starting line. So, this graph gives you the speed of the car anywhere along the track. It gives you, for instance, the speed of the car after 1km, after 1.8km and 2.4km."

Each time an upward motion from the point on the axis to the curve is done but no speeds are read. The presentation is slow.

Questions

1. Can you tell me, from the graph, how many bends there are along the track on which the car was driven?

If the answer is "3", we ask: "Why?". Otherwise, we help the pupils with a series of questions each providing a hint and ask after each one the same question: "How many bends?". Let us give examples:

1a) This graph is not a picture but ... ?
1b) What's the top speed, the slowest speed?
1c) What's the speed at 1km, 2.5km?
1d) How many times has the car been slowed down?

In order to prepare the ground for Part C, the following questions are asked:

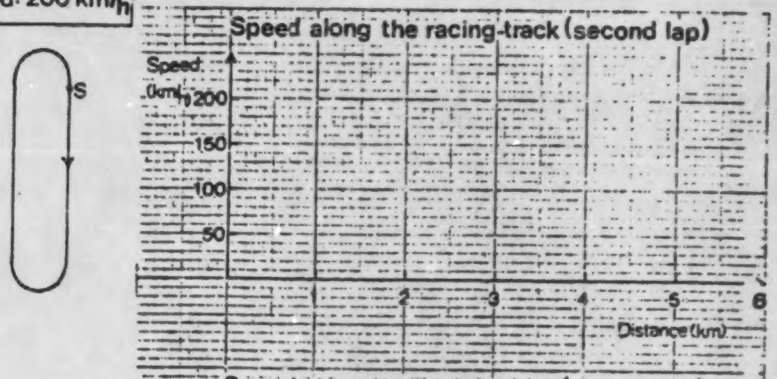
2. Which bend is the worst? the easiest? the "second worst"?

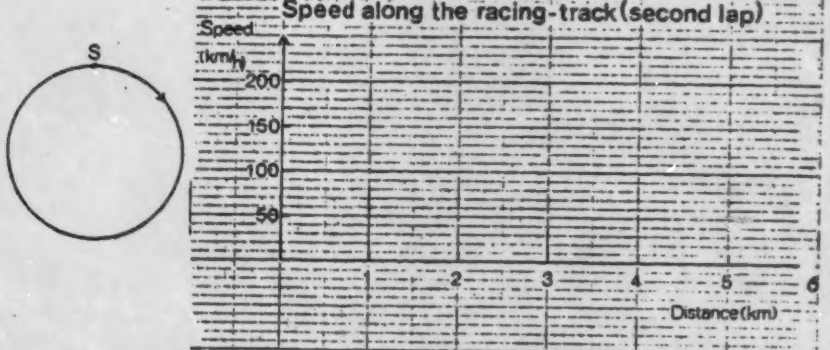
Part B

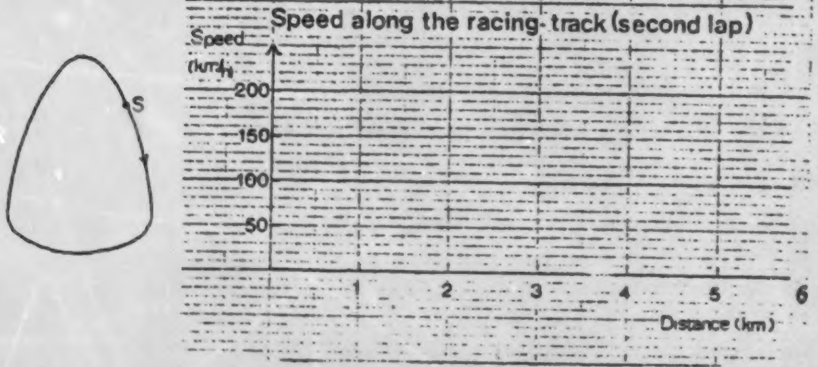
Sheet 1.1 is brought in replacing graph 1.1 which is slightly set aside. The pupil is invited to sketch for each track appearing on the left-hand side a sensible speed-graph similar to the one they have worked on. Each track is shown in turn with a comment that each is six kilometres long. The graph paper already scaled is presented slowly with an emphasis on each axis, but mainly on the horizontal one. The numerals, 1, 2, 3, 4, 5 and 6 are read and a comment

Sketching a speed graph

Top speed: 200 km/h


S
Speed along the racing-track (second lap)
Speed (km/h)
200
150
100
50
1
2
3
4
5
6
Distance (km)


S
Speed along the racing-track (second lap)
Speed (km/h)
200
150
100
50
1
2
3
4
5
6
Distance (km)


S
Speed along the racing-track (second lap)
Speed (km/h)
200
150
100
50
1
2
3
4
5
6
Distance (km)

concerning the length of the tracks is made. The top speed, 200km/h, already appearing on the page, is pointed out. Before the pupil gets down to work, the idea of sketching the speed graph for the second leg is explained. More precisely, the motion of the car around the track for the first lap is simulated with the tip of a pencil and it is stressed that the sketching or the recording "starts" when the car goes by the starting point after coming out of the "corner". Graph 1.1 is removed before the pupil starts.

Part C - Selection of the "right track"

Finally, graph 1.1 is brought back and sheet 1.2 is introduced. The pupil is told that sheet 1.2 shows a selection of tracks on which was driven the car which "produced" graph 1.1. He is asked to make a choice after having been told that only one is correct. The problem finally amounts to making a series of rejections. Since each pupil at first points out one track after more or less thinking, we ask for the ones already rejected and afterwards proceed through a certain pattern which is not rigid.

RESULTS AND COMMENTS

Preliminary comments

The readers must bear in mind that racing cars do not have the same appeal for boys and girls (for cultural reasons which we shall not discuss here). Most English boys frequently play with "Scalextric", a racing kit including miniature remote controlled cars and various patterns of tracks. Also, sport television programmes feature frequent grand prix.

Part A - Results

The main mistake consisted in confusing the graph with the track and consequently pupils answered 6, 8 or 9 bends. Contrary to what happened in the 1st year, in the 3rd and 4th year only a minority made this mistake. The most fascinating finding was the simultaneous use of the graph at the symbolic and pictorial levels. In other words, the graph could be read correctly by many (to find top speed, slowest speed ...) and used right afterwards as a picture of the track. For instance, in first year (written test) 12 girls out of 24 went through all the hints and answer to at least one or two correctly without giving finally the right answer. This was the case for only 2 boys out of 15.

Part A - Interpretation

Interviews showed that familiarity with cars' performance helped some boys (but not all) to use the graph at the proper symbolic level while they were confused with other graphs. Even a far below average boy in 1st year answered quickly "3" showing a clear understanding in his explanations. As for the girls, those with less developed graphical abilities (the case in 1st year) ran into serious trouble. We can easily hypothesize from their answers that they could only count on little situational support to help them to decipher the graph.

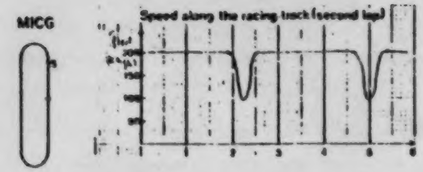
Part B - Results

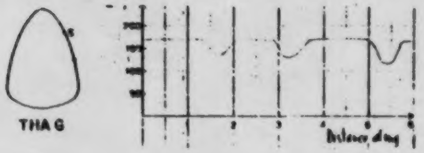
A fair number of pupils drew their graphs while moving their forefingers round the track. This sort of simulation of the car around the track was no more than a plain consequence of the presentation "showing" the car driven around the track for the first lap and coming out of the last bend to start its second lap. One can assume that almost all used some form of visual simulation.

Let us note that this question is far more complex than the former one. Sketching the speed-graph involves a complex coordination between the graph and the track. We shall not enter into a fine description of our classification but rather simply describe 3 categories which could be used as a basis to characterise every answer.

First category:

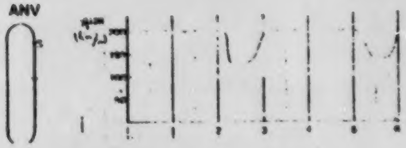
An absence of familiarity with racing cars led naturally pupils to refer in his/her answer to the model provided in Part A. Consequently, almost all the girls treated the question by observing the geometric properties of the tracks. They then spaced more or less correctly troughs (⌣) of various depths. Examples are given below.

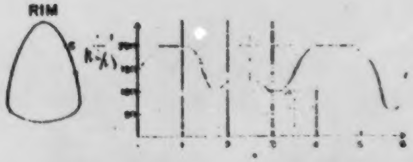




Second category:

For the pupils (mostly boys) who were more familiar with racing cars' performances, situation aspects were equally used such as: constant speed (different from top speed) along curvy stretches, speed at starting line

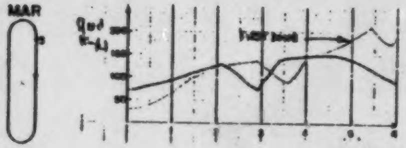




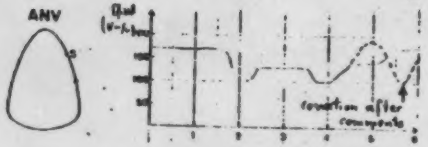
different from top speed, constant speed in certain circular bends ... Sometimes those addidtional elements are correctly integrated with the geometric features of the tracks. (See examples ANV and RIM above.)

Third category:

However, sometimes the additional situational information interferes with the preliminary basic form the graph ought to have because of geometric reasoning. A few pupils appear to have performed a direct simulation of the car's motion through an almost mechnaicl movement of the pencil going up and down as the eyes or the forefingers were going round the track. A comment to the effect that initial and final speeds differ prompted MAR to "adjust" his sketch.



ANV "forgot" the last bend as he was too much concerned with situational details.



Part B - Comments

The situation which helped many boys in Part A now makes the task much harder. Indeed, for many boys, the vivid memories and/or strong mental images which support their thinking conflict with the more basic abstract aspects of the problem. The necessary abstraction is made more difficult because of the larger basis of information from which it is derived.

Pinning down "the" right answer is no easy question. After all, MAR's graph certainly reflects his technique of driving his scalextric cars. Neither is the job of evaluating each performance if one takes into account the strategies used which determine the intrinsic difficulty of the task. We have in fact provided in Part B nice examples of how a task can be made more demanding when a situation is intensively called into play. The results of the next part are also illuminating in that respect.

Part C - Results

The last question of Part A suggests the first criterion to use in the selection. Nevertheless, coordinating a graph and several tracks turned out to be very demanding. In fact, on the one hand, the graph has to be considered symbolically : for instance, pictorial resemblance with elements of tracks has to be disregarded. On the other hand, the tracks (picture) has to be examined at a certain abstract level. At any stage, the process may be disrupted by both factors (described above). We shall not focus our attention on how this coordination is achieved but rather on a few features which are related with the issues addressed in this paper.

a) A category of answers in the written test* has shown a difficulty in scaling up the pictures of the track.

THOG: "The bends are too steep to go round anyway".
CARG: "Because the bends are too close together".

b) During the interviews, we noticed that many pupils were unable to get a relative perception of one bend with respect to the others or to get a total "grasp" of a track. For instance, ANP and MAR would "drive mentally" a car in each bend and determine a sort of "absolute" difficulty for each bend. Written answers similarly suggest such an absence of abstract reading.

FIN: "The bends are not very sharp turning".
DUNG: "The bend is too dangerous, it is the top one and very hard to get around and would slow down speed".

c) Even though a pupil can analyse each bend in relation with the others, each track has to be examined individually since the tracks are not drawn to the same scale. At this stage, pupils tend to compare the difficulty of a particular bend with the one of an adjacent track.

d) Non-symbolic responses were also frequent. It is however impossible to draw any conclusion from the selection of tracks G or B since, each time, pupils were easily turned away from their choice by a short (verbal or written) comment.

But, the association of a straight stretch with a straight portion of the graph is more significant. For one first year girl, track A had to be rejected on the basis that a curvy stretch could not be associated with straight segment of the graph. Further questions showed that the underlying car performances could not be appreciated.

Clearly, on the other hand, this curvy stretch brought about another sort of difficulty to bright boys who insisted that a car could not be driven at full speed on such a "straight" as the graph shows.

Part C - Comments

Once again, we have observed that situations can be used in several ways and help or hinder the coordination between the graph and the tracks. This part also illustrates the fact that situations involve abstract elements (here the picture of the tracks) and conventional aspects (more particularly the last case) which calls for a certain flexibility in assessing the answers.

CONCLUSIONS

The results have shown to what extent each pupil has his/her own situation. We have observed sharply delineated differences between boys' and girls' responses to which we should not as such pay too much attention. The main point exemplified should be the remarkable diversity of personal perceptions and/or conceptions which for each situation brings in a more subtle and complex categorisation of the pupils. This is what we discoversd with most of our graphs. For instance, for pupils living near a harbor the tide may "go up and down" while for "inland" children it comes in and goes out.

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*Note that there was no Part B

In fact, the personalisation involved in each situation brings in various interpretations of the goals of a task and a range of strategies which determine its level of complexity. From such conclusions, didactical consequences should be drawn.

There is an understanding among educators to the effect that the utilisation of situations should mainly help pupils with average academic abilities*. Let us be a little more theoretical. We can assume that a concept is rich in so far as it has a strong basis of abstraction. The more numerous are the conflicting elements and the noise "against" which a concept has to be constructed, the stronger the concept is. However, we must distinguish the construction of concepts from the use of situations in order to make abstract ideas more concrete or, in other words, as a concrete support for abstract reasoning. In fact, the utilisation of situations involved several forms of abstraction even in the "material" presentation of the problem (picture of tracks). We have been simply reminded that science and mathematics make use of situations stripped of many factors, in other words, of schematised situations. Abstraction is a key word. And it would be unwise to equate the ability to abstract with the construction of a larger experimental basis to help abstraction.

If situations are meant to help pupils with average academic abilities, their role should be mainly the development of their capacity to abstract. Such development does not depend on the use of an abstract vocabulary but on a slow and gradual "drawing-out" (abs-trahere) of the main features guided by some precise intentions. Abstraction being a personal "démarche" we are urged to proceed very cautiously in curriculum design. The prospection phase should be progressive and based on discussions, model works, films, experiments ... Many side-issues are of great importance not only to keep motivation high but also to dissipate biased personal conceptions. To haste and create artificial contexts is the main pitfall which we must avoid.

To what extent should situations be used is still an open question. However, we claim that our results point towards some principles as to how they should be used. They are in line with FREUDENTHAL's* phenomenological approach which as reflected in the programme WISKOBAS** promote the utilisation of large scale situations involving weeks of work and the effort to stress the child's point of view more than the mathematical structures.

This is equally true of U.S.M.E.S. rationale which uses the pupils' involvement in real problem solving to make sure that the personal démarche of abstraction takes place. The Spanish GRUP ZERO also adopts positions close to ours in their Matematicas de Bachillerato*** for the Secondary level.

FINAL NOTE

The reader may think before the conference about the training which would improve pupils' performances in the problems described in this paper.

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* of course, the following arguments apply to everyone

* See STREFLAND, L. (1977)
** See Five Years IOWO.
*** See bibliography.

CHILDREN'S OPERATIONAL THINKING WITHIN THE CONTEXT OF BRACKETING AND THE ORDER OF OPERATIONS

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a) Introduction

In a "teaching experiment" (Menchinskaya, 1969) investigating the construction of meaning for the concept of first degree equations in one unknown (Herscovics and Kieran, 1979), we have worked with six students at the junior high school level. We have found that their conception of the meaning of the equality symbol had not evolved from the one found in elementary school children (Ginsburg, 1977). When asked to give an example of an equation, each one provided us with a single arithmetic operation on one side and the result on the other. To expand their notion of equality, we asked them to provide us with examples involving first one operation on each side, and then multiple operations. In order to restrict the word "equation" for use in the algebraic sense, we coined the words "arithmetic identity" to describe the arithmetic use of the equal sign. Our students could readily justify the terminology since both sides of the equality yielded the same value. Following this, the students were asked to operate on their already-constructed arithmetic identities by performing the same operation on both sides.

In constructing arithmetic identities and operating on them, problems involving the order of operations and the use of brackets occurred constantly. In the individual interviews with our six subjects we have found that students view a string of operations, such as 2 + 3 x 4, operationally, that is in a left-to-right sequence. They do not see any confusion in the evaluation of such expressions. Hence the conventional order of operations and the use of brackets seem superfluous and therefore all these rules must be learned "instrumentally" (Skemp, 1976). Thus, although the usual presentation may be thought to be "relational" in that the need for such rules seems to be justified, such is not the case from the learner's point of view. In this paper we examine the student's difficulties with bracketing and the order of operations, and suggest an approach which uses his operational thinking to create a necessary cognitive disequilibrium which may thereby help him see the need for bracketing.

b) Statement of the Problem

Every one of our six subjects had been taught in class, at some time prior to working with us, the use of bracketing and the conventional order of operations. At first glance, a reading of the excerpts below would indicate that some students had forgotten the rules, others had either mislearned them originally or were remembering them incorrectly. However, we discovered that the root of the problem lies much deeper.

The standard textbook presentation of this topic shows the need for some convention by introducing a question believed to be ambiguous, such as, "What is the value of 2 + 5 x 4?" The student is asked, "Is it 28? Is it 22?" He is then informed that "mathematicians have agreed on the following steps to simplify such expressions,"[1] according to the following order:

1. perform the bracketed operations;
2. perform the multiplications and divisions in order from left to right;
3. finally, do the additions and subtractions in order from left to right.

The student is then asked to evaluate strings of operations, such as, 5 + (4 − 1) + 2 x 3. Though he may very well be able to do these exercises at the time, we have evidence to suggest that these conventional rules for the order of operations run counter to the student's way of thinking. Thus it should come as no surprise that he seems to forget or discard these "rules without reasons."

We discovered during the teaching experiment, while our subjects were constructing and operating on arithmetic identities, that they were thinking operationally. For example, our students would not see 2 + 3 as the same operation as 3 + 2. Though the result is the same, in the first case one starts with 2 and adds on 3, which is not the same as beginning with 3 and adding on 2. Thus the written sequencing of their operations in a certain order reflects the order in which they are thinking about these operations. Furthermore, their evaluation of such a string of operations (i.e., 4 + 3 x 12 + 4 + 2) begins with the first operation on the left and continues sequentially from left to right until the last operation is performed to yield a result of 23. This left-to-right tendency also reflects our cultural tradition in writing and reading.

In constructing their arithmetic identities (of which each side is a string of operations), our subjects were writing them down, operation by operation, as they were thinking of them, and were keeping a running total as they went along. They were not thinking of the conventions while constructing or evaluating their own strings. To then be asked to evaluate them according to the conventions conflicted with the more natural tendency of evaluating them in the order in which they were written, which was the order in which they had been conceived. In fact, the evaluation according to the conventions, at times, yielded a value different from what the student had intended. Whenever this occurred, we suggested bracketing certain operations in order not to violate the established conventions.

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[1] M. P. Dolciani and W. Wooton, Modern Algebra, Book 1 (Boston: Houghton Mifflin Co., 1970), p. 41.

In the excerpts below, we shall examine the difficulties which our subjects experienced with bracketing and the order of operations, (1) while they were constructing arithmetic identities and (2) while they were operating on arithmetic identities. The problems encountered while operating on arithmetic identities were of a slightly different nature from those met while constructing arithmetic identities.

c) While Constructing Arithmetic Identities

This section is subdivided according to the following three topics: i) the tendency of our subjects to evaluate their arithmetic identities in a left-to-right order, ii) the effect of this tendency on bracketing, iii) other individual notions on the use of brackets.

i) The Tendency to Evaluate Arithmetic Identities in a Left-to-Right Order

While our subjects were constructing arithmetic identities, we often asked them to give the value of one side or the other. It soon became obvious that all of them were evaluating their arithmetic identities in a left-to-right sequence. This practice, which manifested itself repeatedly, is illustrated in the following two excerpts which are indicative of the thinking of our six subjects.

With Greg:

G: 5 + 2 x 3 =
I: What is the value of that left side?
G: 21.
I: So you're adding the 5 and 2 first, are you?
G: Yes.

With Michel:

I: Would you write down an arithmetic identity?
M: 5 + 10 ÷ 3 = 4 x 2 - 3.
I: What is being done first on this side (left side)?
Is it the 5 + 10?
M: Yes.

When asked to evaluate a side of their arithmetic identity, all responded with a value arrived at by calculating from left to right. In fact, they were writing down their arithmetic identities, one operation at a time, as they were thinking of them, keeping track of the total as they went along. Though they had all worked in the past with bracketed expressions, even at elementary school, none of them used brackets in constructing their arithmetic identities. Only one student, Greg, remembered the conventional order of operations, but this was after he had first done a left-to-right calculation:

I: So you're adding the 5 and 2 first, are you?
(5 + 2 x 3 = . . .)
G: Yes . . . Oh! you have to do the multiplication first.
5 + 2 x 3 = 1 x 10 + 1.

When we noticed that none of our students were following the conventional order of operations (except Greg), we suggested the use of brackets to overcome this problem.

ii) Some Effects of the Left-to-Right Tendency on Bracketing

Because the tendency to evaluate an arithmetic identity from left to right was so predominant, our suggestion to Michel to bracket the operation which appeared first seemed an unnecessary one to him. Michel didn't see the need of bracketing the first operation on the left of each side since these were the ones he was calculating first anyway. Though he acquiesced to our suggestion, as seen in the first excerpt below, he did not use brackets in any successive arithmetic identities except when asked for them, as seen in the second excerpt below:

I: What is being done first on this side (left side)?
(5 + 10 ÷ 3 = 4 x 2 - 3). Is it the 5 + 10?
M: Yes.
I: So, do you want to put that in brackets?
M: (5 + 10) ÷ 3 = (4 x 2) - 3.

And during another session:

I: Could you give me an example where you don't have just one number, the result, on this side?
M: 3 + 23 x 2 = 26 x 2.
I: What is the value of the left side?
M: 52.
- - -
I: Since 3 + 23 has to be added first, you put it in brackets.
M: (3 + 23) x 2 = 26 x 2.

Though Michel didn't see that brackets made any difference in the two examples above, he seemed more inclined to see their usefulness in the next example:

I: Would you write down another arithmetic identity?
M: 3 + 1 x 4 = 10 ÷ 2 + 2.
I: What is the left side?
M: . . . 7 . . . (but seemed puzzled).
I: What do you want to do first?
M: 1 x 4.
I: Since you want to do that first, let's put it in brackets.
M: 3 + (1 x 4) = 10 ÷ 2 + 2.

It's obvious why Michel seemed puzzled. The tendency to evaluate his written arithmetic identity in a left-to-right order was very strong. What had happened here was that Michel had thought of an arithmetic identity with a left side of "3 + 4". However, since we had been encouraging examples with multiple operations, he had written down 3 + 1 x 4 = . . . Thus when he was asked to evaluate the left side of 3 + 1 x 4 = 10 ÷ 2 + 2, there was a conflict between the intended value of "7" and the value of "16" arrived at by his left-to-right method. Therefore, bracketing for this kind of example, where one wished to keep intact some operation other than the first one, seemed to make sense to Michel.

The influence of the left-to-right tendency on bracketing was seen also with Caroline. We did not ask her about bracketing an operation which appeared at the extreme left of either side of the arithmetic identity. However, when questioned about the bracketing of some operation other than the first one, Caroline not only inserted brackets around it but also rearranged the arithmetic identity so that the bracketed operation appeared first.

C: 4 x 3 + 1 - 3 = 3 x 2 + 4.
I: Have you learned the use of brackets yet?
C: Yes.
I: What if you wanted to add 3 + 1 first?
C: (3 + 1)4 = (2 + 2)4.

Caroline's approach implied that whatever operation was to be done first should be placed first at the extreme left of each side. We also noticed that Caroline changed not only the right side, but also dropped the "- 3" from the left side of the arithmetic identity when doing her "rearranging."

Patricia was another student who seemed to think that whatever was bracketed was not only to be done first but also to be placed first. The following sequence indicates what took place prior to our first question about bracketing:

I: Could you give me an example where you have two operations on each side, but they're not both the same?
- - -
Pa: 2 + 3 x 5 = 3 x 5 + 2.

We observed that this particular example of Patricia's was different from several of her previous ones in that it involved commutativity. To continue:

I: What is the value of the left side?
Pa: 17.
I: What are you doing first?
Pa: I'm adding. No, I'm multiplying. No, . . .

A conflict was noticed here (very similar to Michel's with the arithmetic identity, 3 + 1 x 4 = 10 ÷ 2 + 2) between what was intended, "17", and the tendency to evaluate the left side from left to right. Patricia was becoming confused. To continue:

I: Have you learned the use of brackets?
Pa: Yeh, I have. (2 + 3) x 5 = 3 x 5 + 2.

She immediately put brackets around "2 + 3", the first operation, without any thinking about the effect on her arithmetic identity. We had thought that she might bracket "3 x 5" to maintain her arithmetic identity.

I: So that's 2 + 3 which is?
Pa: 5.
I: And 5 times 5?
Pa: 25.
I: O.K. So, what do we have over here (right side)?
Pa: . . . That's wrong.
- - -
I: Why do you put brackets?
Pa: To show which one you do first.

Thus even though she had wanted to do "3 x 5" first, the tendency to evaluate the written arithmetic identity from left to right and the notion that it is the first operation which is bracketed caused her to bracket "2 + 3" rather than "3 x 5."

We find in most elementary algebra textbooks the rule: "Anything that is bracketed must be done first." Caroline's and Patricia's examples are clear indications of how this can be misinterpreted. In their examples they show that the bracketed operation must not only be done first but also appear first. Other children, such as Michel, do not make this mistake. In fact, they do not see the need of bracketing until they construct an arithmetic identity which when evaluated (from left to right) conflicts with their mental construct.

We have seen the effect of the left-to-right tendency in the evaluation of an arithmetic identity and its subsequent influence on bracketing. We now examine some other ideas which our subjects had on bracketing.

iii) Other Bracketing Ideas

In this section we will observe some unconventional uses of bracketing by three of our subjects, Caroline, Barbara, and Piero.

In addition to the uses already cited, Caroline sometimes employed brackets to set off one side of an arithmetic identity (or equation) from another:

I: Let's try one more equation where you have two operations on each side, with an unknown hiding one of the numbers.
C: With brackets?
I: If you'd like.
C: (6 x 5 + 1) = (a x 5 + 1).

In another different type of example, Caroline used brackets to indicate alternate ways of expressing the same number, 2 x 3 + 4 + (2 x 10) = 2 x 4 + 2 + (4 x 5). Here she was not implying that these bracketed operations should be done first, otherwise they would appear first (as has been shown before). She was rather drawing attention to two different "replacements" of the same number, "20," in her arithmetic identity.

Barbara used brackets in a somewhat similar sense. She constructed several of her arithmetic identities by taking a number from the left side and expressing it as a bracketed product on the right side:

I: Make an arithmetic identity now, please.
B: 72 x 3 = (9 x 8) x 3.

Piero constructed his arithmetic identities without brackets and evaluated them from left to right, just as our other subjects had done. However, when asked about the use of brackets, he remembered some "old" rules:

I: How about an example with different operations and different numbers?
Pi: 2 x 2 + 4 = 8 - 6 + 6.
I: Have you learned the use of brackets?
Pi: Yes.
I: Can you give me an example where you use brackets?
Pi: 2 + (1 x 3) - 2 = (1 x 3) - 2 + 2.

And in another session:

I: So if we divide the right side by 2?
Pi: 6 x 4 ÷ 2 = 24 ÷ 2 . . . Shouldn't we put brackets around, (6 x 4) - 2 = 24 ÷ 2?
I: Why?
Pi: Because then you might get mixed up in . . . well, in this case, it doesn't really matter because the multiplication is before division. But in cases like "7 + (3 x 3)" you should always use the brackets.
I: To show which operation you want to be done first?
Pi: Yes.

In learning the rule in school that multiplications and divisions are done before additions and subtractions, Piero had somehow picked up that multiplications take precedence over divisions. Furthermore he had also mysteriously learned to associate bracketing with multiplication. For Piero, not only did multiplication take precedence, but it was also bracketed. In those examples where a multiplication occurred first, one was free to bracket or not. Piero was still evaluating his arithmetic identities from left to right, but was ensuring that any multiplication was kept intact by the use of brackets.

Thus we can see from the examples of these three students that they will construct a variety of rules which may or may not be erroneous.

d) While Operating on Arithmetic Identities

The previous section described the bracketing difficulties encountered by our subjects while they were constructing arithmetic identities. This present section will describe the bracketing difficulties encountered in a different context. Here the students were operating on their already-constructed arithmetic identities. Most of our subjects merely tacked on the new operation at the right end of each side, without using brackets. In doing so, they were consistent with their own intuitive conventions of calculating from left to right.

However, two of our subjects, Patricia and Michel, in doing this "tacking on," experienced two different types of difficulties. Patricia's problem occurred as soon as she was confronted with the insertion of a second operation. In a previous example, 12 x 11 = 132, when asked to add 2 to the left side, she had responded with "Oh, 14!", having added 2 to 12. The same confusion is evident in the following excerpt:

I: What happens if we subtract 2 from the right side (10 - 7 = 3)?
Pa: - - - 10 - 7 = 3 - 2.
I: How is that an arithmetic identity?
Pa: No.
- - -
I: Alright, what can I do, using subtraction, to make it an arithmetic identity?
Pa: - - - Minus 2.
I: You'd like to subtract 2?
Pa: Yeh, but you can't take 2 away from 10.

Patricia was focusing on the first number appearing on the left rather than the expression on the left. Therefore, we suggested that she put brackets around 10 - 7 to show that it was the original operation on the left side. Although in the conventional sense the use of brackets is unnecessary in this example, nevertheless it helped her. In fact, she continued on her own to insert brackets in subsequent examples and did not experience again the difficulty described above.

With Michel, we were the prime source of his confusion. We made a pedagogical mistake which brought to the foreground the possible effects of violating the student's natural thinking:

I: Let's take 4 + 2 = 9 - 3. Multiply the right side by 2.
M: 4 + 2 = 9 - 3 x 2.
I: You can put the 9 - 3 in brackets. Then it's clear which operation is done first. 4 + 2 = (9 - 3)2.

As has already been pointed out, Michel's left-to-right convention would not require bracketing in this type of example. To continue:

I: Is it an arithmetic identity now?
M: No.
I: What can we do to make it an arithmetic identity again?
- - -
M: Times 2. Do I put it in front?

This question may have been prompted very simply by the lack of space between "4 + 2" and the equal sign. However, since Michel was using brackets in this example, there didn't seem to be any need to write the multiplier after the brackets. To continue:

I: Yes, you could. It doesn't matter if you put it in front or in back. Let's put it in front then as you suggested.
M: 2(4 + 2) = (9 - 3)2.

But this was not a good idea, as the excerpt below will indicate. We should have maintained a left-to-right sequencing of operations, corresponding to Michel's natural tendencies. The blunder becomes obvious in an example where we were not interjecting with the use of brackets:

I: Take this arithmetic identity (4 x 12 = 48 - 0) and build a new one from it.
M: Plus 2? 2 + 4 x 12 = 48 - 0.
I: Is that still an arithmetic identity?
M: No. Put 2 on that side. 2 + 4 x 12 = 48 - 0 + 2.

Michel had added on to the left side by placing the "add on" at the front (following the same pattern as was set in the previous excerpt). Although he didn't notice the resulting discrepancy in this example, he did in the next one:

I: On the line below 4 ÷ 1 = 10 - 6, show me how you build a new arithmetic identity by doing some operation on 4 ÷ 1 = 10 - 6.
M: 4 x 4 ÷ 1 = 4 x 10 - 6. It won't go.

Michel had multiplied both sides by 4 and had placed the 4 at the beginning of each side. However, a quick evaluation (according to his left-to-right convention) made it clear that "It won't go." We then reminded him about bracketing the original operation on each side. Though this suggestion enabled him to balance his arithmetic identity, it was not getting at the root of Michel's difficulties. Tacking on at the beginning not only went against his method of evaluating, but also created for him notational problems:

I: I would like you to build a new arithmetic identity from this one, (5 + 10) ÷ 3 = (4 x 2) - 3.
M: + 7(5 + 10) ÷ 3 = + 7(4 x 2) - 3.

Note how Michel has added 7 on both sides by preceding the numeral with the operation symbol. This is possibly the best example one can find to illustrate the operational nature of his thinking.

I: Now what does the + 7 in front of the bracket mean to you?
M: Plus 7.
I: Oh, you're adding 7, are you?
M: After.
I: Are you adding 7 and then dividing the whole thing by 3, or do you want to do all of this [(5 + 10) ÷ 3] first, and then add the 7?
M: Add the 7 after.
I: O.K.
M: Do I do it after?

We corrected this problem by directing Michel to add on 7 at the end of each side and also suggested a second set of brackets to show that the original expressions were to be evaluated before performing the new operation, [(5 + 10) ÷ 3] + 7 = [(4 + 2) - 3] + 7. A week after this session, Michel was operating on his arithmetic identities by placing the new operation to be performed at the right end of each side. Furthermore, he was continuing to construct and operate on arithmetic identities without bracketing, and evaluating them according to his left-to-right convention.

e) Summary, Conclusions, Suggestions

We have shown in this section the difficulties which our subjects experienced with bracketing and the conventional order of operations. This problem, uncovered within the context of our teaching experiment, proved to be much greater than expected. Though we proposed bracketing certain operations in some of their arithmetic identities to avoid contravening the conventional order of operations, our students (except Greg) never seemed to be aware of any conflict and had to be shown every time such a conflict occurred.

Because our subjects were working with their own strings of arithmetic operations, while focusing on the construction of arithmetic identities, we were able to assess their natural tendency of evaluating strings in the order in which they were written, which was exactly the same as the order in which they had been conceived. As a result, the conventional order of operations (which they had all learned at some prior time in school) seemed quite arbitrary and unnatural, and therefore was easily forgotten (as indicated by most of our subjects) or remembered incorrectly (as indicated by Piero).

Since our students wrote down their arithmetic identities, operation by operation, as they were thinking of them, this had an effect on the way they operated on them. From our work with all our subjects, we found that they have a natural tendency to perform additional operations in the order in which they are conceived. The problems caused by not following this natural order were especially evident in the excerpts with Michel [e.g., $+ 7(5 + 10) \div 3 = + 7(4 \times 2) - 3$].

The use of bracketing, when proposed to some of our students, indicated problems associated with their interpretation of the rule that "whatever is bracketed should be done first." For Caroline and Patricia, bracketed operations in arithmetic identities were not only to be done first, but also to appear first. By way of contrast, other students, such as Michel, saw no need at all for bracketing the first operation. This was the operation they would be performing first in any case, since it reflected the beginning of the sequence of their mentally-constructed operations. Such students could see a need for bracketing only when some operation other than the first happened to be a written replacement for their mental construct of a single number, as seen in the case of Michel's "$3 + (1 \times 4) = 10 \div 2 + 2$."

These findings explain why standard textbook presentations on bracketing and the order of operations have little effect on many students whose left-to-right tendency prevents them from seeing any ambiguity in the evaluation of $2 + 5 \times 4$. Our work suggests a more compelling approach to the topic of bracketing. By using an arithmetic identity (such as $2 \times 5 = 10$) rather than a string of operations (such as $2 + 5 \times 4$), one can effectively establish an awareness of the need for bracketing. Taking $2 \times 5 = 10$ and subsequently replacing 5 by $4 + 1$ yields $2 \times 4 + 1 = 10$. Since the left side must still have a value of 10, the child, whether he evaluates by his own left-to-right method or by the standard ordering conventions, will come to see that the only way to maintain this arithmetic identity is by bracketing, hence $2 \times (4 + 1) = 10$. It is only by creating in the student's mind a *need* for such a notation that he will accept it and thereby make use of it. For if "relational understanding" means knowing which rule to use and why it works (Skemp, 1976), the student must first be aware of the necessity of such a rule.

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Constructing Meaning for First-Degree Equations in One Unknown

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This study addressed itself to the development of a teaching-learning scheme for first-degree equations in one unknown and to an examination of the way in which students think about and understand the concepts involved, during the actual teaching process. The methodology used was a version of the Soviet "teaching experiment" which involved two 25-45 minute individual interviews (audio-taped) with six students of various abilities from grades 7 and 8 (aged 12 - 14 years).

Summary of teaching-learning scheme

The teaching-learning scheme, based on learning theories of Piaget, Steffe, Herscovics, and Bruner, aimed at 1) constructing meaning for the concept of equation, and 2) laying the groundwork for justification of the algebraic operations used in the solution of equations. This scheme is not meant to replace standard presentations, but rather be used as an introduction to the topic of equations at the junior high school level.

In order to have the student construct meaning for the new mathematical form, using as a starting point his already-acquired arithmetic knowledge, it was first necessary to expand his notion of the equal sign. This was done by having the student construct arithmetic identities (e.g., $6 \times 4 + 3 = 29 - 2$) with several operations on each side. Thus he no longer saw the equal sign as being a link between an operation on the left side and the result on the right, but rather as a sign of the equilibrium between two sides "having the same value".

The next step, going from arithmetic identity to equation, included both a change in content and a change in form. In order to lead the student gradually to the acquisition of meaning for the new form of an equation, Bruner's three modes of representation were used. A number in one of the student's arithmetic identities was hidden first by a finger (enactive), then by a box (iconic), and finally by a letter (symbolic). Thus an equation was defined as an arithmetic identity with a hidden number. In following these three stages the student was able to acquire an intuitive understanding of the concept and then gradually transform this to a formal understanding.

Arithmetic identities were also used to establish the justification of the rule ("Whatever you do to one side, do to the other side.") which would be used in the solution of equations (solution processes, however, remain beyond the scope of this study). By performing an operation on one side of an arithmetic identity, the student was able to see the need for performing an operation on the other side to restore the balance. When restricted to the use of the same operation, he was led to doing the same thing on both sides.

Experimental conclusions

This section will present some of the thinking patterns which emerged during the individual interviews with the six students.

(a) Pretest

The object of the pretest was to determine the student's existing ideas on equations and the equal sign.

For five of our six students, their notion of the equal sign was restricted to a single operation on the left side and the result on the right. Their thinking in this regard seemed to be similar to that found at the elementary level where, according to Ginsburg (1977), children look at equalities in such an operational way. This pointed out the need for expanding the student's notion of the equal sign to include arithmetic equalities containing several operations on both left and right sides simultaneously.

Some of our students had difficulty in verbalizing what they meant by "equation" and the equal sign. They didn't know how to say "it", but could give examples. This confirmed Laborde's (1978) findings that junior high school students have a tendency to define "dynamically", that is, they give an operational explanation of the event with the subevents leading up to it described in the order in which they occur, and usually by means of specific examples.

(b) Extending the notion of the equal sign

The notion of the equal sign was extended to include multiple operations on both sides. This expanded class of equalities was given the name "arithmetic identity", a term which reflected both the arithmetic nature of these equalities and the identical value borne by both sides.

When asked for an equality with an operation on each side, three of the four subjects questioned (the other two subjects had been *shown* such arithmetic identities) responded with an example involving commutativity. When asked for an equality with a different operation on each side, two students thought initially that they had to use the same numbers on both sides. Our six students were able to go on to construct arithmetic identities with multiple operations on each side, the "value" of each side being the criterion for equality.

In these constructions all our subjects wrote down their arithmetic identities, operation by operation, as they were thinking of them, keeping track of the total as they went along. Thus they viewed their arithmetic identities sequentially (a sequence of operations which yielded a certain value) rather than globally (each side as "another name for" the same number).

(c) The concept of equation

By means of a three-step representational process, arithmetic identities were transformed into equations. A number in an arithmetic identity was hidden first by a finger, then by a box, then by a letter. In this way an equation was defined as an arithmetic identity with a hidden number.

Our six students could readily find justification for the letter being called an "unknown". From their responses, we also noticed that three subjects (one of whom had seen equations in class) referred to the notion of solving equations. Even though we hadn't spoken about solving equations or even the need to solve equations, it seemed that some of our students felt that this was, in some way, an essential part of working with equations. As a matter of fact, it was quite natural for them, even at the construction-of-equations stage, to look at an equation and mentally try to slot in the required number. Their answers indicated not only that they had the notion of uncovering (i.e., solving), but also that this uncovering would bring back the arithmetic identity. This awareness seemed to occur quite naturally. They had acquired the notion of solution as a result of the way they had constructed meaning for equations. Thus our students realized that not only could they go from the arithmetic identity to the equation, but also from the equation back to the arithmetic identity (if they knew the hidden number).

It is quite interesting to note that, when asked to build equations, five of our six subjects did not follow the sequence of our construction, i.e., starting with a written arithmetic identity and then hiding a number, but instead wrote equations immediately (the sixth student was *instructed* to write the arithmetic identity first). This seemed to indicate that the concept of equation had acquired sufficient meaning to allow our students to skip the intermediary step.

In looking for common misconceptions in their interpretations we noticed that all our students were able to "conserve equation" (Wagner, 1977), and all (except one who had been taught equations in school) avoided restricting the unknown to any one side.

(d) Operating on arithmetic identities

We did not use the scale to induce the general rule ("Do the same thing to both sides") because of its limitation to addition and subtraction of natural numbers. Arithmetic identities were used instead, for these mathematical representations of equilibrium were not restricted by the physical limitations of the scale. Furthermore, any operation performed on an arithmetic identity was immediately verifiable by the student. Thus, since our students now defined equations as arithmetic identities with a hidden number, the operations performed on arithmetic identities could be eventually transferred to operations on equations.

In trying to induce the rule with the first two subjects, our questioning was found to be too vague, leading them to perform different operations on each side. Since we wanted our students to perform the same operation on each side, we restricted our line of questioning with the four remaining subjects in the following way: after adding a number to one side of an arithmetic identity, we asked them, "*Using only addition*, is there anything that you can do to make it an arithmetic identity again?"

Our four remaining subjects were able to induce the addition rule (Add the same thing to both sides), some of them requiring more examples than others. However, it was the ease with which three of them were able to induce equivalent rules for the other operations that indicated the effect of our revised line of questioning. One of our weaker students needed a few more examples for subtraction and multiplication, but became as efficient as the others for the last operation, division.

It was but an easy step for our four students to reach the next level of abstraction, that of considering the individual rules as examples of the general rule.

This experiment was designed to investigate whether or not the constructions involved were accessible to students representing a wide range of ability. From the written and verbal evidence of our six subjects, we could conclude that all of them were able to construct meaning for each of the main concepts: extending the meaning of the equal sign; constructing equations; operating on arithmetic identities.

This novel, yet simple, approach can also be used effectively with older students for remedial purposes, as shown by a follow-up study with a class of mature adults taking high school algebra at university.

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THE "DETERMINATION" IN THE LANGUAGE USED IN THE MATHEMATICS CLASS-ROOM

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A study of mathematics texts, and in particular, of mathematics schoolbooks, shows an important usage of the natural language beside that of other means of communication (such as diagrams, graphs, or symbolic notation). When speaking, in the classroom interchanges are also made to a large extent in the natural language, and help to acquire the concepts and to facilitate the learning of notions. Also, what the pupil says, translates in part his conceptions, his manner of taking hold of a certain concept.

This is the reason, it seems important to analyse the forms taken by the language in the teaching of mathematics (of the manual, of the teacher in the classroom, or of the pupil) and to analyse the differences which can exist between the expression of the pupil and that of the teacher (or of the school-book).

One of the aspects which gives rise in part to large differences is that of "determination". Briefly, determination is the process in discourse which allows one to locate a noun with respect to other elements in the statement, or with respect to the situation of the production of the discourse

e.g. "the figures placed to the left of the point indicate the whole part of the decimal number" (a French schoolbook for the sixth grade - 11-12 years old).

- the participal phrase "placed to the left of the point" allows one to make precise what are the figures in question by situating their with respect to something given in the preceding text : the point of the decimal number.
- in the same way a determination (adjective followed by a noun complement) operates on the noun "part".

I - Importance of determination in mathematical discourse :

One can easily see determination when one reads mathematics texts, even for young pupils. In fact, mathematical objects are often complex objects wich can depend on other objects ; thus a circle can be defined by its radius and its centre ; a point by the intersection of two straight lines ; a set by a property of its objects.

These dependence relations are translated by procedures of syntax such as :

- chains of noun complements ;
- chaining of adjectives or of participles themselves commanding propositional groups.

The following example combines the two procedures : "the foot of the perpendicular directed from the point A to the straight line xy".
The structure can be represented as follows : N_0 of [N_1 Adj [from N_2 to N_3] .

Determination intervenes also on the level of determinants (the, a, each, every...) whose usage in mathematics is very precise because it corresponds to that of quantifiers. In particular, the use of the definite article "the" and of the indefinite article "a" does not correspond in all cases to current usage (especially in French)

II - In schoolbooks :

Determination takes place in the language of schoolbooks as it does in other mathematical works of a higher level. It occurs in cascades of noun complements, or of adjectives governing other complements, all ofwhich is not always easy to read for the pupils even in more "concrete" situations, such as this exercise from a sixth grade schoolbook :
"given that the volume of cider is proportional to the mass of apples, what is the coefficient of proportionality of the volume of cider to the mass of apples when 360 kg of apples give rise to 225 ℓ of cider".

It is important to notice that determination as it occurs in texts is made with respect to terms in the text, that is, with respect to mathematical objects and not with respect to the situation of the production of the discourse that is not with respect to the author or to the moment or place when he is writing.

III - In classroom :

Three French sixth grade classes, with three different teachers are been recorded during several sessions in different situations : work done in groups, explanations of the teacher and collective responses of the class.

1) Importance of the situation of the production of the discourse :

This is the most noteworthy aspect of those of oral determination. The reference points are occasionally other elements of the statement (especially in discourses of the teacher) but above all it is the situation of the production of the discourse (the class, the diagram on the blackboard, the teacher, the pupil) wich appears at the level of characteristics.

Let us cite the most frequent :

- "ce", "cette" ("this") makes reference much less to an object already cited than to an object materialized by a diagram, a piece of writing, etc.
- "celui-ci", "ce...là" ("this one here", "that one there", "that...there") accentuates even more the role of the materialized reference point.
- "ça" ("that"), wich does'nt occur at all in texts, appears a lot when speaking, especially in the case of pupils.

Examples :

Teacher :	"Your neighbour for example, look at the shape he has made for me. Is it exact ? Is it that form ?"	"Bon alors, votre camarade par exemple, voilà la forme qu'il m'a faite. Bon heu, est-ce que c'est exact, est-ce que c'est cette forme ?"
A pupil :	"Not exactly that"	"Pas tout à fait ça"
Teacher :	"Yes, it's rather..."	"Oui, c'est plutôt"
A pupil :	"That one there, yes that one"	"Celle-là, oui celle-là"
Teacher :	"This one here"	"Celle-ci."

Finally, the presence of personal touches, wich is also specific to the speaker language, allows one to locate the object with respect to the two speakers. It takes place as much in the discourse of the teacher as in that of the pupil.
"Where are your points situated at 4 cm ?" (Teacher to a pupil)
"I count 4 cm and I join my circle" (A pupil to another pupil).

These determination features in the oral discourse give another status to the mathematical object, or give it a directly accessible point of reference (on the black board, in the notebook) or on indirectly accessible one by means of a person who is present ("my" circle, "your" points).

2) Variations according to the situation :

In our recordings, we can distinguish two situations :

a) the teacher in front of the whole class : the frequency of "celui-ci" "celui-là" and especially of "ça" is lower. One can find some chains of noun plus noun complements.

b) the teacher speaking individually with a pupil or with a small group of pupils. The frequency of the "ça", "celui-là"... is much higher.
Teacher : "c'est quoi là ? ça c'est quoi ?" (What's that there ? That what is it ?)
Teacher : "Et pourquoi t'as fait ça là ?" (and why did you but that there ?)
The inherent ambiguity in the "ça" is lower on account of the restricted number of speakers.

In conclusion, it would seem that the features of determination are strongly dependent on the situation of production of the discourse, at least on the side of the teacher. In fact, we have been able to observe that is not completly the case in the discourse of pupils.

IV - The pupils :

In the recordings of the pupil exchange, independently of the situation, we have been able to notice how strong the reference to the present situation is :

"ça va comme ça" (That works like this)
"c'est ça" (That's right)
"Alors, ça c'est B" (So, that's B)

and that reference to other elements of discourse is weak.

Now, in differents groups of exercise papers of pupils, we were able to observe the same phenomenon ; the determination was done as in oral discourse ; a lack of clarty or an ambiguity thus resulted in different situations of wich the following are examples :

- use of the definite articles ("the") before a noun which nothing in the statement determines (nor in what precedes, nor in what follows by the use of noun complements)
  e.g. "the number", "the circle" (when it is question of several numbers or several circles).

- use of "ce" "cette" in the same conditions

- use of "ceci" "cela" "ça" which refer to something vague or to a procedure, a way of doing something.
  "Ca fait toujours des cercles" (That always makes circles)
  "Cela fera toujours pareil dans tous les cas" (That will always do the same in all cases).

- chain of complements determining the noun is incomplete
  e.g. "All even numbers of multiples of 3"
  instead of
  "All even numbers from the list of multiples of 3".

The operation of determination in discourse corresponds to an operation on mathematical objects ; it corresponds to the establishment of relations between different things and to structuring these relations. The difficulties for the pupil in handling determination in discourse can result from difficulties in these mental operations on mathematical objects. To help himself, the pupil needs implicit references (as in oral discourse), which appear in his expression in writing or in speaking.

Generalising as an aspect of mathematical thinking

Chris Laenen

In the psychological as well as in the didactic literature, generalizing is brought to the fore as a fundamental aspect of thinking. It is even so that a lot of mathematicians and scientists consider generalising as one of the most important basic components of mathematical thinking.

Nevertheless, in the published works we find no unanimity about the exact definition of the specific character of the generalisation process. Besides, most of the descriptions of generalisation are very confused, which often give rise to different interpretations. Yet a distinction must be made between generalised thinking, which, in psychology, is being directly connected with the problem of "transfer" and is, as an elementary thinking activity, lying at the basis of other mental actions, and on the other hand generalising as a specific mathematical action. In the first case, generalising refers to the capacity to apply a known way of solution to a new problem after recognising similar situations.

We illustrate with a simple example:

Suppose that a pupil has learned that a difference of two squares can be separated in factors with the formula:

$a^2-b^2 = (a-b)(a+b)$, with the help of the following examples:

$$x^2-4 = (x-2)(x+2)$$
$$x^2-64 = (x-8)(x+8)$$
$$x^2-3 = (x-\sqrt{3})(x+\sqrt{3})$$

If now the pupil succeeds in separating a given polynomial $4x^2-y^2$ into $(2x-y)(2x+y)$ after discovering the similar form between the given polynomial and the polynomials in the examples, then, from the psychological point of view, the pupil has generalised.

It is in this sense that we must interpret the generalising activity in the works of KRUTETSKIJ (1) and BARTLETT (2).

In mathematics the act of generalising is very specific and has to be distinguished in a didactical perspective from generalising in the abovementioned psychological sense or in a lot of other senses as we find in DIENES (1), MARJORAM (4), WHEELER (5), RUBENSTEJN (6), VIGOTSKIJ (6), GERKESOV (6), DAVYDOV (6), CHAPMAN, (7), TALL (8), BAUER (9).

Similarly in mathematical education, where we aim at the development of the fundamental mathematical actions (10), actions like extending, continuing and generalising must be distinguished from each other so that at the right moments learning processes can be organised to learn to use these actions consciously. (For the notion of a mathematical (mental) action we join with the point of view of VERMANDEL (10).)

It is my intention in this paper to contribute to this aim. Generalising as a mathematical action is characterised by the fact that is leads to a new mathematical entity: a generalisation. The attainment of this result guarantees at the same time that the action is completely and successfully finished.

To generalise

In mathematics, the situations in which we generalise can be separated into three classes. We distinguish: generalising structures, generalising definitions, generalising theorems. From the examples it will be clear that the generalisation process in each of the three cases is characterised by the same thinking skill.

a) generalising structures

1) Let us suppose that the pupils have the knowledge of the set $\mathbb{Z}$ of integers and the ordered structure on the set $\mathbb{N}$ of the natural numbers. This ordered structure $(\mathbb{N}, \leqslant, A_1, A_2, A_3)$ is characterised by the set of axioms:

$$A_1: \forall\, x \in \mathbb{N}: x \leqslant x$$

$$A_2: \forall\, x,y \in \mathbb{N}: x \leqslant y \text{ and } y < x \Rightarrow x = y$$

$$A_3: \forall\, x,y,z \in \mathbb{N}: x \leqslant y \text{ and } y \leqslant z \Rightarrow x \leqslant z.$$

With this structure as a starting point, we can ask the question: can we define on the set $\mathbb{Z}$, which includes $\mathbb{N}$ as a subset, a similar structure $(\mathbb{Z}, R', A_1', A_2', A_3')$ which includes the given structure as a substructure ?

In this problem the pupils are stimulated consciously to an action, of which we postulate the purpose.

The first stage of the real action consists of constructing an ordered structure on $\mathbb{Z}$ of which we suppose that it will meet the requirements.

The construction of the order-relation can be attended with a trial-and-error process. Let us assume that someone suggests that an order relation R' on $\mathbb{Z}$ can be defined as follows:

$$R' = \{(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+;\ x \leqslant y\} \cup \mathbb{Z}^- \times \mathbb{Z}^+ \cup \{(x,y) \in \mathbb{Z}^- \times \mathbb{Z}^-;\ -y \leqslant -x\}$$

in which $\leqslant$ is the natural order relation in $\mathbb{N}$.

It is very easy to verify that R' satisfies the properties of an order-relation:

$$A_1': \forall\, x \in \mathbb{Z}: xR'x$$

$$A_2': \forall\, x,y \in \mathbb{Z}: xR'y \text{ and } yR'x \Rightarrow x = y$$

$$A_3': \forall\, x,y,z \in \mathbb{Z}: xR'y \text{ and } yR'z \Rightarrow xR'z.$$

The structures $(\mathbb{Z}, R', A_1', A_2', A_3')$ and $(\mathbb{N}, \leqslant, A_1, A_2, A_3)$ are similar. In the following phase of the action we must show that $(\mathbb{N}, \leqslant, A_1, A_2, A_3)$ is a substructure of $(\mathbb{Z}, R', A_1', A_2', A_3')$. In this case it is easy to verify that indeed $R'_{|\mathbb{N}} = \leqslant$ and $(A'_{1|\mathbb{N}}, A'_{2|\mathbb{N}}, A'_{3|\mathbb{N}}) = (A_1, A_2, A_3)$.
The action is finished.

We shall call "generalising" the mental process that systematically includes the following phases:

- postulating the purpose of the action: defining on $\mathbb{Z}$ an ordered structure which includes the given structure as a substructure. (The formulation of a purpose means that a certain action must be performed consciously.)
- the constructing of a structure on $\mathbb{Z}$, similar to the given structure, and which is probably a possible construction.
- proving that the constructed structure on $\mathbb{Z}$ includes the given structure as a substructure.

We say "the structure $(\mathbb{N}, \leqslant, A_1, A_2, A_3)$ has been generalised to $\mathbb{Z}$".
The result of the generalisation process, the structure $(\mathbb{Z}, R', A_1', A_2', A_3')$ shall be called a "generalisation" of the given structure $(\mathbb{N}, R, A_1, A_2, A_3)$.

2) Let us suppose that the topological space (A, T_A) is given in which $A = \{1,2,3,4\}$ and $T_A = \{\emptyset, A, \{1\}, \{2\}, \{1,2\}\}$ is a topology on A.
The topological space (A, T_A) is characterised by four axioms. Then we can ask the pupils the following question:
Can we define on $B = \{1,2,3,4,5,6\}$ $(A \subset B)$ a topology T_B in such a way that the given structure (A, T_A) is a topological subspace of the topological space (B, T_B)?(*)

The purpose of the action is fixed. In the second phase the pupils have to construct a topology T_B on B which meets the requirements. If, for instance, T_B is defined as $T_B = \{\emptyset, B, \{1,2,5\}, \{1,5\}, \{2,5\}, \{5\}\}$, then T_B is a topology on B. Afterwards they have to verify the equation $T_{B|A} = T_A$, which is very trivial in this case. So the topological space (A, T_A) is a substructure of (B, T_B). We call the performed action "generalising".

More generally, if a structure S' on a set A is given and $A \subset B$, then we call generalising the mental process characterised by:

- the formulation of the purpose: can we define on B a structure S' which is similar to S and which includes S as a substructure ?
- constructing a structure S' on B, similar to S, and which includes probably S as a substructure.
- proving or verifying that S is a substructure of S'.

In general we can define a generalisation of a structure as follows:
We call a structure S_1 a generalisation of a given structure S_2 if and only

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(*) If the pupils are already familiar with the term "generalising" then we can formulate the problem as follows: "Can we generalise the structure (A, T_A) to B?"

if S_2 is a substructure of S_1.

We illustrate this definition with some examples:

1) the vector space $\mathbb{R}, \mathbb{R}, +$ is a generalisation of the vector space $\mathbb{R}, \mathbb{Q}, +$.

2) the topological space $(\overline{\mathbb{R}}, \mathcal{T}_{gew})$ is a generalisation of the topological space $(\mathbb{R}, \mathcal{T}_{gew})$.

So we can establish (and this is not a mathematical action) that a structure S_1 is a generalisation of a given structure S_2, without generalising.

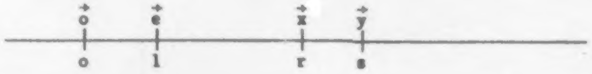
b) Generalising definitions

A defninition that refers to a certain class of objects can be a starting point to generalise.

If D_1 is a definition in relation to a class A of objects, then we call the action by which we create consciously a definition D_2 for a class $B \subset A$ of objects which includes the given definition as a particular case, generalising.

In that case definition D_2 is called a generalisation of definition D_1.

Example: the scalar product in Π_o can be defined by a generalisation process. If the pupils know the definition of the scalar product of two vectors lying on the same vector line.



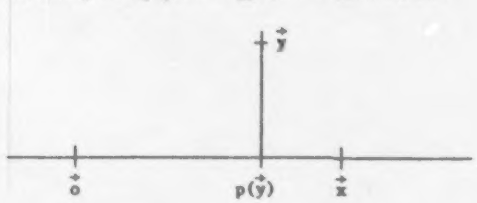
$$\vec{x}.\vec{y} = r.s.$$

then we can ask the pupils the following question :

How can we generalise the definition of the scalar product of two vectors to Π_o ?

In other words: how can we define the scalar product $\vec{x}.\vec{y} \in \mathbb{R}$ for two vectors in Π_o so that for two vectors on the same vector line, the definition conicides with the previous one ?

If, for instance, the pupils suggest the definition:



$$\vec{x}.\vec{y} = \vec{x}.p(\vec{y})$$

then they have generalised the definition from which they started.

c) Generalising theorems

If P is a theorem in relation to a class A of objects, then we call the action by which we consciously postulate a property Q for a class $B \supset A$ of objects and which includes P as a particular case, generalising.
The the theorem Q is called a generalisation of theorem P.

The generalising of a theorem is an action which is characterised by the same phases as the generalising of structures.

Let us suppose for instance, that the pupils know the definition of the scalar product in Π_o and that they have proved the following theorem for the class of vectors of the same vector line D :

$$\begin{aligned} \vec{x},\vec{y},\vec{z} \in D_o : \quad a,b \in \mathbb{R} : \vec{x}.(a\vec{y}+b\vec{z}) &= a.\vec{x}.\vec{y}+b.\vec{x}.\vec{z} \\ (a\vec{x}+b\vec{y}).\vec{z} &= a.\vec{x}.\vec{y}+b.\vec{y}.\vec{z}. \end{aligned}$$

Then we can ask the following question: Can we generalise the previous theorem to Π_o ? This means: can we formulate a property for the elements of Π_o which includes the previous one as a particular case ? In this case it is evident that the required generalisation, if it exists, must be an analogous property.

The last phase of the generalisation process exists in proving the sentence

$$\begin{aligned} \vec{x},\vec{y},\vec{z} \in \Pi_o, \quad a,b \in \mathbb{R} : \vec{x}.(a\vec{y}+b\vec{z}) &= a.\vec{x}.\vec{y}+b.\vec{x}.\vec{z} \\ (a\vec{x}+b\vec{y}).\vec{z} &= a.\vec{x}.\vec{z}+b.\vec{y}.\vec{z}. \end{aligned}$$

To extend

To define extending a mathematical action we shall make a distinction between extending a set and extending a function.

In short, extending a set A means that, in order to find a solution for a given concrete problem, new elements are added to the set A so that a set B is being created that includes A as a subset.

We shall call the constructed set B an extension of set A.

The extending of a set is a mathematical action which, in the course of the ages, has evoked great heights in the history of mathematics. for instance, the introduction of rational and real numbers, the introduction of complex numbers, ... in mathematics, are results of extending a previously known set.

In secondary education we can regularly create situations in which the pupils experience the necessity of extending a given set and in which they really will achieve this action to the extent that it is possible.

We'll immediately give some examples:

1) In the first form of secondary education, the set $\mathbb{Z}$ of integers

can be introduced by extending the set ℕ of natural numbers.
Indeed, the pupils experience that there are pairs of natural numbers for which subtraction is not defined. This gives rise to the following problem: how can we extend the set ℕ or, which elements must we add to the set ℕ, to give sense to the difference of each pair of natural numbers and in such a way that this difference belongs to the new set ?
The answer is simple: by defining for each natural number the opposite $-n = 0-n$, the set ℕ can be extended.
This extension $\mathbb{Z} = \{0,1,-1,2,-2,...\}$ is called the set of the integers in which now subtraction too is internal.
2) The extending of the set of real numbers for the pupose of computing a square root for each real number leads to the introduction of the set of complex numbers. Although it is rather improbable that the pupils will suggest a solution for this extension problem completely on their own, it is still important to treat this problem as a typical extension problem with them, as an example to illustrate the essential character of the action of extending. It is important to stress the fact that only if the constructed set comes up to the previously defined requirements, the action may be called extending.

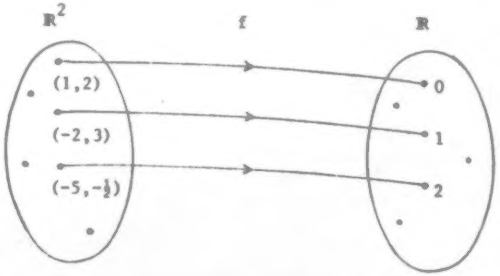
The notion "extension of a set" has also its own specific meaning. For instance we can state that a set B is an extension of a set A, if A is a subset of B and if B has a property which is not present for the elements of set A.

For instance we can call set $B = \{1,i,-1,-i\}$ an extension of set $A = \{1,i\}$ because $A \subset B$ and because B is closed under multiplication while A is not.

In the same manner we shall call a function F an extension of a given function f if $f \subset F$ and if F has a property which is not characteristic for a given function f.

Extending a function is an action by which one constructs for a given function an extension in relation to a previously defined property.

1) function f:



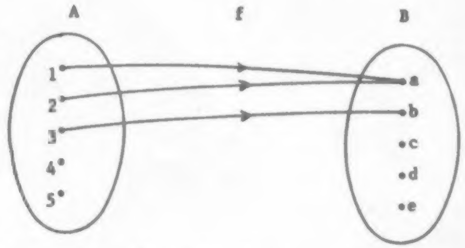
can be extended to a symmetrical function from $\mathbb{R}^2$ to ℝ by defining

$$f((2,1)) = 0,\ f((3,-2)) = 1,\ f((\tfrac{1}{2},-5)) = 2.$$

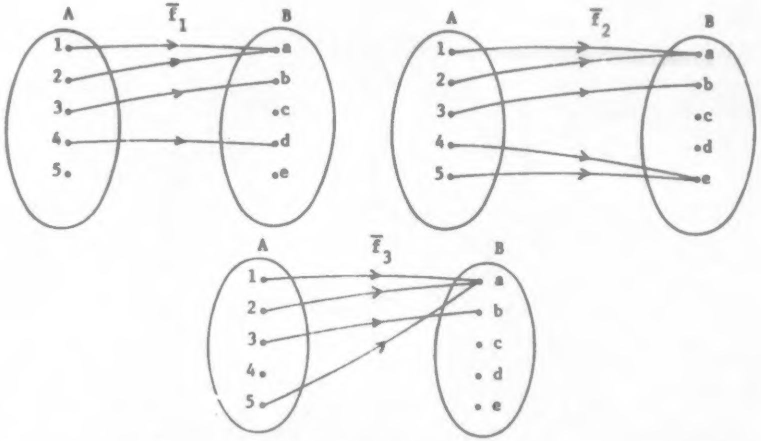
2) The function f: from ℝ to ℝ : $x \mapsto 2x(x-1)/(x-1)$ is not linear, but can be extended to a linear function $\overline{f} = f \cup \{(1,2)\}$.

In reality, situations that can be created in which the pupils experience the necessity and sense of extending given functions are rather rare. Generalising functions, on the contrary, is a fundamental mathematical action which can be frequently operated in the intruction of mathematics. Still it is importatnt to bring to the fore the distinction between the extending of functions and the generalising of functions, because on the whole the terminology is rather confusing. It is also necessary to introduce a third related notion, that of continuing functions.

Suppose that a function f from A to B is given by the arrow representation:



Then we can continue the function f in different manners to a new function $\overline{f}$ that includes f as a subset.



The functions $\overline{f}_1$, $\overline{f}_2$, $\overline{f}_3$ are called "continuations" of the given function f.

In particular we shall call $\overline{f}_1$ a continuation of f to point 4.

Definition: If f is a function from A to B then we shall call $\overline{f}$ a continuation of f if and only if $\overline{f}$ is a function and $f \subset \overline{f}$.

To continue a function means that a new function is constructed which includes the given function as a subset.

We can continue a function, but we can also restrict a given function by leaving out a number of pairs. Each subset of a given function is called a restriction of this function. Continuing functions and restricting functions are inverse actions.

In the same way we can, instead of extending a given function, look after restrictions of a function which have a certain property that is not characteristic for the given function. This action, the inverse one of extending, shall be called "reducing"; the result of this action shall be called a reduction.

As opposed to the extending of functions, in secondary education we can regularly create situations in which the ability of the pupils to reduce functions lies at the basis of solving certain problems. We illustrate this with a concrete example.

Let us suppose that the pupils have knowledge of the goniometric functions

sin: $\mathbb{R} \to [-1,1]: x \mapsto \sin x$

cos: $\mathbb{R} \to [-1,1]: x \mapsto \cos x$

tg : *from* $\mathbb{R}$ *to* $\mathbb{R} : x \mapsto \operatorname{tg} x$

and that they can construct their graphs.

As an introduction of cyclometric functions we can pose the following problem to the pupils: since $\sin: \mathbb{R} \to [-1,1] : x \mapsto \sin x$ is a surjective map but not injective, the inverse relation is not a map. How can we restrict the domain of the sine so that this restriction is a bijective map ?

Here pupils are confronted with the inverse problem to extending. One must search for a restriction of the given function which has one property more than the given one.

The pupils can deduce the solution of the problem form the graph of the sine function. The reduction of the function according to the requirement can have different results. Each result is called a reduction of the given function.

In an educational system that wants to make the general development of the pupils' intellectual capacity its prime aim, actions such as generalising and extending are important cognitive capacities that certainly belong to the objectives of mathematics. In this option, mathematical didactics will demand of the teacher a selection of the subject matter and an organisation of the learning processes in such a way that the pupils will be encouraged to generalise as much as possible. The examples given form concrete classroom practice prove that the subject matter of current instruction already offers a certain number of possibilities to teach and exercise the actions of generalising and extending. It is clear that pupils won't always be able to arrive independently at the correct solutions of a generalising problem. In this case it is the teacher's task to organise the learning processes in such a way that the pupil's participation remains optimal.

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Social/Affective Factors Influencing Problem Solving Capabilities

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The excerpts below represent subsections of a longer paper about "applied mathematical problem solving processes" which was presented at the 1979 Warwick meeting of the IGPME.

Good mathematical problem solvers are people who interpret many events in mathematical ways and who enjoy using their problem solving capabilities to find out more about the situations. Or, in real problem solving situations, the critical difference between a good and poor problem solver frequently consists of the ability or inclination to convert an ouch into a well defined problem. Minimally, a well formed problem must involve: (a) a global conception of what a final solution might be--at least in the sense that one can determine whether or not the problem has been solved, (b) a hope of solving the problem--including the realization that one's problem solving abilities are relevant to the situation, and (c) a desire or willingness to solve the problem.

Good problem solvers are able to make early and accurate assessments about the relative difficulty of problems--about how long they will take to solve, and about what sorts of resources will be needed. These initial judgments about time constraints or about problem difficulty often radically influence the type of problem solving strategies students will use in given situations--and they are closely related to assessments of personal competence.

In multi-stage problems or projects, good problem solvers are often somewhat like good runners in their abilities to pace themselves during prolonged problem solving efforts. That is, they know when to relax and when to exhaust themselves; they know when to work out meticulous details and when an "incubation period" (Poincare, 1948) might be helpful. For example, in recent research at Northwestern University (Cardone, 1977) in which groups of four students worked together on project-type problems in sessions that lasted two hours each, "good" problem solving groups were compared with less productive groups (where productivity was judged on the basis of both quantity and quality of work produced). In these setting, it became clear that the good groups were not the ones characterized by dogged determination--never "goofing off," never taking breaks, never getting off task. On the contrary, clear work cycles were evident in the behavior of the more productive groups. At predictable points in the cycle, good groups took breaks involving jokes or lighthearted discussions about topics unrelated to the problem (e.g., movies, dates, campus activities, etc.)--or, in some cases, members of the group would go to the Coke machine, to a rest room, or to a quieter place in the hallway. Furthermore, when the instructor prevented the good groups from "goofing off" at strategic points in the work cycle, overall performance deteriorated in terms of both quality and quantity. On the other hand, if the instructor monitored the work cycle of less productive groups, then productivity could be increased through imposed "breaks" or incubation periods at strategic points in the cycle.

The above sorts of phenomena have occurred in a number of pilot studies involving a variety of two-hour group problem solving projects. More closely controlled studies are currently being conducted to identify causes for the phenomena, conditions for its occurrence, and factors related to its appearance. Tentative explanations involve two types of factors: (1) In well functioning groups, individuals gradually develop the belief that the group is more productive than they would be if they were working as individuals. In these groups, individuals intuitively learn to engage in activities that seem to be designed to improve the cohesive functioning of the group (more than to improve the productivity of isolated individuals within the group).

The development of group personalities, and of roles for individuals within the groups, is an important aspect of group problem solving. Furthermore, the development of group personalities gives valuable insights into the development of problem solving personalities for individuals (Bales, 1955). Roles and personalities change to fit various stages in the problem solving process, and some stages (e.g., incubation periods or breaks) seem designed to facilitate role changes. In fact, knowledge about stages in problem solving seems to enhance problem solving ability precisely because it creates a reliable set of cues for critical role changes. A specific example is given under (2) below.

(2) Good problem solving individuals, and good problem solving groups, persevere at solution attempts when other individuals and groups give up. But, perseverance is not simply a function of initial motivation to solve the problem. It is greatly influenced by a learned sense of cumulativeness in prolonged solution attempts. To a good problem solver, the path to the solution of a problem involves various stages--with each building upon the previous stage; and a major factor that enables a good problem solver to persevere is a sense of how far he or she is along the path.

Heightened anxiety often accompanies critical stages in which "breakthroughs" in problem solving may occur. So, if a student does not have a sense of where these anxiety crises fit into the overall problem solving process, and if the student does not have a sense of cumulativeness in the efforts he or she is making, s/he may "give up" completely rather than (for example) take a brief break to muster all of his/her resources to meet a critical stage.

When educators and researchers talk about problem solving situations, they often ignore the fact that most people work on real world problems when other people and other resources are available. People seldom work in isolation using only the power of their own minds to solve problems. Instead, good problem solvers learn to amplify their own powers through effective use of outside resources. For example, when real people solve real problems, one of the most often used problem solving strategies is to "ask someone who can give the needed information." This is not to say that good problem solvers solve most problems by asking someone else to do their work for them. Recognizing situations which fit mathematical models and formulating a problem in such a way that a specific bit of information can be requested are two of the most obvious distinguishing characteristics of good problem solvers. They are not trivial skills.

When a psychologist gathers information about students solving artificial problems in isolated laboratory situations, the data are usually analyzed using a computer. The psychologist does not use the computer because he is too ignorant to calculate the relevant statistics with a pencil and paper; he uses the computer because the important aspects of his own problem solving situation, are to select and interpret the appropriate calculations--not to actually go through the drudgery of computation. Furthermore, to decide upon the most appropriate computations to make, the psychologist probably also consulted a statistician before he carried out his analysis. Again, this does not mean that the psychologist is ignorant about statistics; to formulate the problem in such a way that a statistician can help required a firm understanding of statistics and research design as well as a powerful stock of conceptual models and structural metaphors.

The above example involves three basic types of "outside" resources: (1) conceptual models (including an appropriate language and representational system) borrowed from some discipline, (2) other people, (3) tools (e.g., computers, calculators).

When problem solving is carried on in isolation, many useful problem solving processes have been neglected. Neglected processes include common skills involving information transmission or reception, and they also include processes associated with "brainstorming," or other group problem solving procedures (e.g., synectics, Gordon, 1961; Prince, 1968--bionics, Papanek, 1969, etc.) that have been used effectively for years in business and government, but which have received minimal attention from educational research or instruction.

One of the attributes that makes group problem solving interesting is that, in many respects, group problem solving is analogous to individual problem solving (Gordon, 1961). Groups, like individuals, develop over time, and the stages that a group goes through in its development are similar to the stages an individual goes through. However, groups have the advantage that many processes which are implicit, internal, and quite difficult to master in individual problem solving settings often are explicit, external, and considerably more simple and meaningful in group settings. When internal processes are externalized in the context of small group activities, their development is often easier to trace in clinical research settings and they are often easier to describe in a form that is understandable to average ability problem solvers.

Two examples will be given below to illustrate how difficult processes may be simplified in small group settings. The examples are not intended to imply that we should explicitly teach group problem solving techniques. Rather, group problem solving situations furnish an effective context in which to investigate and teach individual problem solving procedures--especially when the problem solver is at a relatively primitive level of skill acquisition.

(1) Problem solving strategies like "identify the givens," "identify the unknowns," "eliminate irrelevant information," or "organize the information given," all fall under the general heading "understand the problem." Unfortunately, the suggestion to "understand the problem" seems rather useless to poor problem solvers. What does this mean apart from the more specific directions like "identify the givens," "identify the unknowns," or "eliminate irrelevant information?" On the other hand, how can one eliminate irrelevant information etc. if one does not understand the problem? In small group problem solving settings, the way out of this dilemma is to give more meaningful instructions like "Use your own words to describe the problem to a friend," or "Describe the problem in such a way that a friend could give you the answer." Poor problem solvers sometimes flounder with a problem for a long time before noticing (if asked) that they are unable to give a clear description of the problem. By trying to describe a problem to a friend, a student is often forced naturally to eliminate irrelevant information, to identify knowns and unknowns, and to "organize the information given."

In a problem solving course for elementary education majors at Northwestern University, students sometimes work in pairs, sitting on opposite sides of a partitioned table (see Figure 1). A problem or problem situation is given to one person in each pair--i.e., the sender. It is not the sender's job to solve the problem; rather the sender must describe the problem to her partner so that her partner can solve the problem--or give directions for solving the problem.

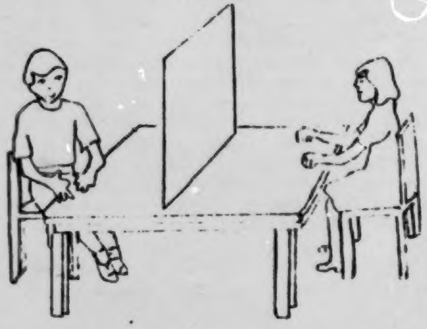

Figure 1

When students work in the above types of situations, or when they work together in small groups to solve problems, they can be encouraged to notice that (a) some people are good talkers while others are good listeners, (b) some people are good generalizers while others are better at working out details and (c) in general, that a variety of different roles are beneficial at different stages in problem solving. Good problem solvers must be flexible enough to switch quickly from one role to another while solving a problem and they must become skillful at selecting roles that are beneficial at particular stages in the problem solving process.

(2) Problem solving strategies like "consider a problem," "consider an auxiliary problem," or "consider a special case," can be summarized with the simple advice, "look for a related problem." Yet, to poor problem solvers, this advice often seems quite foolish because, "I already have one problem I cannot do, I do not need another." To poor problem solvers, a more sensible suggestion is, "Look at the same problem from a different point of view."

In several recent research studies (e.g., Cardone, 1977) where groups of four students were supposed to work together on problems, individuals often sat together but worked independently--each conceptualizing the problem in quite different ways, and each unconscious of misleading biases inherent in his/her point of view. If a student has not yet developed an appropriate conceptual model to fit a given problem situation--or, if the student has not yet developed the relevant intellectual structures, he will generally have the following cognitive characteristics when he is forced to make judgments which rely on the use of this model or structure: (1) centering--i.e., he will not "read out" all of the information that is available; he will focus on only the most obvious features of the situation and will fail to notice less obvious features; (2) egocentrism--i.e., he will "read in" meaning and information because of his own preconceived biases; he will distort the situation to fit his own understanding even when his own ideas do not correspond to objective reality.

To overcome both of the above tendencies (i.e., centering and egocentrism) poor problem solvers may find it helpful to work in groups in which they are forced to coordinate their own point of view with that of others. While one person may center on one aspect of a situation, another person may center on another; and people with various idiosyncratic interpretations of a stituation will be forced to confront one another. This is one reason why "brainstroming" is often a useful problem solving technique. In group brainstorming sessions, students can be bombarded with a variety of different ideas and approaches, and simultaneously can become more self-critical about their own points of view. In problem solving settings, a good problem solver learns to behave as though he were, within himself, several people sitting around a table working together to solve a problem. He is objective in the sense that he sees the problem from multiple perspectives and he is aware of (and self-critical about) his own perspectives at any given moment. In this way he is forced to be more analytic and to attend to more aspects than are implicit in any single point of view.

In group problem solving settings two distinct but closely related types of structures interact to influence problem solving capabilities. These are: (1) the structure of the group and (2) the cognitive structure of each individual relative to the particular problem solving task (Lesh, 1979). Piaget (1971), Dienes (1963, 1969) and various mathematics education researchers have characterized the conceptual models and cognitive structures that are needed for a variety of mathematical tasks, and several social psychologists (e.g., Stein & Goldman, 1978; Asch, 1952) have talked about groups in terms of their structure. In fact, Baldwin (1969), Asch (1952), and Feffer (1970) have talked about the structure of individual personalities (e.g., a person has a "well integrated" personality) and have described how the structure of individual personalities are reflected in group interactions.

Uncoordinated social structures produce behavioral chracteristics similar to those produced by uncoordinated cognitive structures--e.g., centering and egocentrism. For example, when a group is in the early stages of developing a well organized group structure (1) individuals typically notice only the most salient characteristics of their colleagues and themselves--e.g., a student may say, "Sally was talking all the time and was getting all the problems right," when in fact Sally may have been having great difficulty on most of the problems and only talked on a few occasions when she was sure about the correctness of her answers, (2) individuals often project characteristics and attitudes onto other people which they in fact do not have--e.g., a student may say, "Sally thinks she is smarter than the rest of us," when in fact Sally may have had a distinct mathematical inferiority complex.

It is well known that group structures influence individual personality structures (Bales, 1955). For example, group counseling activities are used to help individuals develop stable and healthy personalities (Ohlsen, 1977). And, individual personality structures can influence group structures (Feffer, 1968, 1970)--as, for example, when induced fear or anxiety cause deteriorations in group interaction and functioning. But, more interestingly for the purposes of this paper, both group and individual personality structures influence--and are influenced by--cognitive structures of individuals in particular task situations. For example, in the parts of Cardone's (1977) study that involved group problem solving, when a new and difficult problem was introduced to a well functioning group, the group structure would typically deteriorate temporarily in terms of quantity, quality, and complexity of verbal interaction. On the other hand, if something was done to cause a breakdown in the group structure (e.g., by inducing anxiety through the threat of a test) then it was often possible to noticeably retard learning or problem solving for the task at hand; that is, the quantity and quality of solutions decreases. In the case of individual personality structures, group personality structures, or individual cognitive structures, weak structures were indicated by tendencies to center and to be egocentric. Therefore, these cognitive characteristics can be used as instructional tools to encourage cognitive growth.

In modern psychology it has become more and more clear that cognitive adaptation exists in an ecological system with other adaptation-seeking mechanisms--and that it is influenced by them. Learning, socializing, and adjusting cannot be completely separated. Much that we now call learning is social learning, and many aspects of human learning that we have traditionally regarded as "cognitive" development are specializations of "social" development--including the individual's conception of the self (Mead, 1934) and his moral ideology (Kohlberg, 1969).

In part, learning to be a problem solver means acquiring a problem solving personality. For example, a good problem solver is usually a person who interprets an unusual number of daily situations as problems--that is, as situations where his problem solving skills may be relevant. A good problem solver quickly identifies a new problem as "do-able" or "un-do-able" and next as "easy" or "difficult." Then appropriate solution strategies are selected to fit the initial appraisal. Partly as a result of these characteristics and competency assessments, people who are good problem solvers in one context, in one type of situation, or in one discipline, may be average or below average problem solvers in another.

Psychological descriptions of people have for a long time centered around the premise that people have traits, personality traits, and cognitive traits, that endure and that are manifest in all situations. This notion is currently under strong attack, particularly in personality theory. Research has shown that students form *severe personae* for different situations--family, peer groups, school. The "personality" is different at different times, in different situations, in different mental and physiological states, as a function of cognitive, social, and emotional load, and as a function of the student's particular agenda at the moment. Cognitive performance is moved upward and downward by load factors in the student or in the situations--e.g., noise, emotionality, distraction, confusion, shyness, anxiety. So students fluctuate in their apparent ability depending upon time and place, and "good problem solvers" in a particular area learn to compensate for the above sorts of factors--or to manipulate them to facilitate problem solving efforts.

The above sorts of factors suggest a variety of instructional benefits associated with group problem solving activities. For example, Triandis (Note 1) lists the following points:

(a) Imitation: There is a good deal of learning that takes place when humans see connections between what other people do and rewards they receive. One of the characteristics related to this factor is the viewpoint that behavior is not viewed as being reliably followed by particular outcomes good or bad. Triandis calls this "ecosystem distrust." Seligman (1975) call it "learned helplessness." Rowe (1974) refers to the factor as "locus of control." That is, some students seem to view life in general, and problem solving in particular, as a "crap game" in which success is more a matter of good luck than good management.

(b) Groups create satsifactions with certain activities. An activity which has low intrinsic value when the learner is alone, may acquire a much higher value when the learner is in a group. Group norms, or roles adopted by the learner in a given social setting, or interpersonal agreements between the learner and important others, can change the desirability of activities.

(c) Self-esteem is important. The learner's view of the self, particularly beliefs that certain kinds of learning are possible and desirable, is important in learning. Among learners who think that they are not capable of learning . some material, or who do not see the desirability of learning it, this factor severly reduces learning. A person's self-concept is in part formed through interaction with other people.

Group problem solving sessions can influence a number of cognitive factors in addition to the social/personality factors mentioned above. Furthermore, some of these influences have important implications for both research and instruction in problem solving. For example, many people believe that the best way to learn to solve problems is simply by solving problems--and afterwards, looking back to reconsider processes that were most helpful. Unfortunately, if the problem solving process is not broken down into stages, and if students are forced to work entire problems in isolation, the "looking back" technique seldom works. Average problem solvers seldom solve problems quickly enough to look back on more than a few problems. So, the strategies they notice are often useful for only a small number of problems of a particular type, and they do not learn cues to determine when each strategy might be useful. Furthermore, average problem solvers often have difficulty focusing on the processes that they use to solve problems. Even when they are able to give correct answers, they are notoriously unable to give accurate descriptions of steps in the process they used.

To practice selecting appropriate problem solving procedures for particular tasks, a teacher can give students a whole list of problems and ask them to postpone trying to solve any of the problems until they have first gone through the entire list--trying to determine which technique would be "best" for each problem. For example, for computational problems, students can "set up" each problem before attempting to actually carry out the procedures they selected. Individual students can be assigned to calculate answers for different subsets of the problems, and finally, the group can work together to estimate the reasonableness of the answers they receive. By working in a group, students can investigate critical parts of more problems, and by working on whole sets of problems, one stage at a time, they can focus on "non-answer giving" stages in problem solving. Furthermore, by working with other students, average problem solvers often find it easier to focus on processes that would otherwise have been internal and more difficult to notice.

In most human enterprises, there are meta-level understandings that greatly influence performances on isolated tasks. Some of these understandings have to do with "locus of control" factors or other social/affective understandings; but others have to do with higher order cognitive factors. For example, in problem solving, one meta-level understanding has to do with going beyond the analysis of individual problems to analyze the solution procedures themselves. If a particular problem is treated as just one instance of a general class of problems--some of which will occur at a later time, then the goal may be viewed as more than simply finding a solution to a particular problem; it is also to find a clever/elegant/efficient solution that might be useful later.

When students work in groups to solve problems, they often see that most problems can be solved in a variety of ways--some of which are "better" than others. In research with gifted youngsters (e.g., Krutetskii, 1976) the hallmark of outstanding problem solvers is not so much whether answers are right or wrong but whether "clever" procedures were used. "Good" problem solvers are flexible thinkers who are capable of solving problems in several different ways; so, when one path is blocked, another route can be taken. They are also individuals who actively look for clever and elegant solution procedures.

A second type of meta-level concept has to do with the misconception that mathematics problems should be answerable through brief one-step procedures--rather than through a series of substeps, and perhaps trials and errors. Children learn this misconception quite early in school. Kindergarten is perhaps the last grade in which a teacher can ask a novel question and expect a majority of her students to begin *thinking*. By the time children reach the first grade, they typically respond in one of two ways: (a) they will immediately "know" the answer and will produce it without much thought, or (b) they will stare at the ceiling waiting to "get it" or will say, "I haven't learned that yet." *They are reluctant to think*. They have learned (wrongly) from their teacher role models that good mathematics students do not have to think. Teachers often have a great fear of being asked novel mathematics questions for which they do not immediately know the answer. Consequently, they usually try to avoid or conceal instances in which they actually must think. The role they portray is one modeled after their university professors, who usually appear to students to be "fountains of truth," who always know the correct answers and who are never puzzled, stymied, or confused. The students are seldom allowed to see (even in a pretend, role playing fashion) their professors think. They do not realize that the primary goal of university professors is to lead their graduate students to ask questions whose answers are unknown.

In mathematics classes, students learn a great deal more than mathematics concepts. Minimally, they also learn certain meta-level concepts *about* mathematics and mathematical thinking. Some of these meta-level conceptions or attitudes can facilitate mathematical problem solving abilities, and some can retard it. But, if group problem solving situations are properly managed, students can have an opportunity to witness more than mathematical content, they can also witness mathematical thinking--through peers and through teacher role models.

Group problem solving techniques have been most popular in business and industry, to produce "better answers" by getting input from more people with more areas of expertise. Group problem solving has seldom been used as an instructional setting to improve the problem solving capabilities of individuals, nor has it been used as a research setting in which to investigate processes that are important in individual problem solving.

"Mathematics laboratory" theorists claim that mathematical ideas and processes are acquired through interactions with the environment--and that these interactions are of two basic types: (a) interactions with concrete materials (or lower order "concrete" ideas) and (b) interactions with other people. For example, a teacher can *concretize* a given idea or she can *externalize* the processes underlying the idea. However, neither of these ideas have had significant influences on problem solving research or instruction.

Research on group problem solving is at an extremely primitive state and it is virtually nonexistent in mathematics and science education (Davis, 1973). Furthermore, what little research that has been done has been characterized by obvious shortcomings--e.g., groups have been studied over periods of time insufficient to allow for minimal stabilization of group structure and organization. Nonetheless, because of the many assets described in this section, group settings furnish ideal contexts for research on mathematical problem solving.

Problem solving processes which involve the use of other people as resources are among the most important in realistic problem solving situations. And group problem solving settings seem to furnish powerful techniques for instruction in problem solving. However, a great deal of instructional development and research will be needed before these potentialities can be realized.

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Inductive and axiomatic reasoning at elementary school level

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Summary.

We explained in previous papers how we used the frame of games to introduce a formalism to 8 and 9 year olds. This formalism is non-verbal and is suggestive of a logic. We want to show how such formalism can be used to encourage the development of rational thought. We shall attempt to show that it is thus possible (at elementary school level) to provoke reasoning based on induction and axiomatisation.

Introduction.

The aim of our work was to develop the spirit of synthesis, the spirit of research, the communication inside the group, in children at elementary school. The experiment described here is the final phase of a research program which started in January 77 with a class of 9 year olds (4th grade). Our aim was not to teach something to the students but, having given them a teaching frame, find out what our students could discover, how they did it and how they used it.

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One can thus say that the technique we used (introduction of a non-verbal formalism in the framework of games) apparently had two consequences : firstly it made the student aware of abstraction, secondly it gave the student the urge to abstact, guiding him automatically towards rational thought.

In this paper we describe the work of a 6th grade. We worked with 24 students : in September 78, 8 were 10 years old, 9 were 11 years old and 7 were 12 years old. Before the period referred to, we worked with them for a period of 18 months and we introduced our formalism during their 4th grade (Lowenthal, 1978 ; Lowenthal & Severs, 1979).

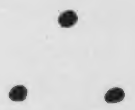
First approach.

A) To study the children's possibilities of abstraction, we gave them, at the beginning of the school year, a set of rules inspired by that used for the Nim game. These rules could be applied in various circumstances. Our aim was to let the children look for the winning strategies and for methods used to discover them. We give here the set of rules we used with our students.

Rules of the game of Adam and Eve : "Adam and Eve see pawns on the blackboard. They play alternately. Adam is always the first to play. If there is at least one pawn left on the blackboard, then the player who must play next can take as many pawns as he wishes among the pawns which are still on the blackboard, but only in one horizontal line. This player must take at least one pawn. The player who is forced to take the last pawn has lost. The game then immediately stops."

A set of pawns, placed on one or more horizontal lines, is called a configuration. The rules can be applied to any configuration : whatever the number of pawns and/or the number of horizontal lines.

Figure 1.



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Inductive and axiomatic reasoning at elementary school level

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R. Severs Instruction Publique, Ville de Bruxelles

Summary.

We explained in previous papers how we used the frame of games to introduce a formalism to 8 and 9 year olds. This formalism is non-verbal and is suggestive of a logic. We want to show how such formalism can be used to encourage the development of rational thought. We shall attempt to show that it is thus possible (at elementary school level) to provoke reasoning based on induction and axiomatisation.

Introduction.

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In this paper we describe the work of a 6th grade. We worked with 24 students : in September 78, 8 were 10 years old, 9 were 11 years old and 7 were 12 years old. Before the period referred to, we worked with them for a period of 18 months and we introduced our formalism during their 4th grade (Lowenthal, 1978 ; Lowenthal & Severs, 1979).

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A) To study the children's possibilities of abstraction, we gave them, at the beginning of the school year, a set of rules inspired by that used for the Nim game. These rules could be applied in various circumstances. Our aim was to let the children look for the winning strategies and for methods used to discover them. We give here the set of rules we used with our students.

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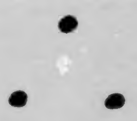
Figure 1.



Adam can win the game for the configuration shown in figure 1 : he simply starts and take both pawns on the second line.

B) System of representation used.

A given configuration will be represented by an n-tuple. So (4, 27, 86) represents a configuration with three horizontal lines and 117 pawns : 4 on one of the horizontal lines, 27 on another and 86 on the third one.

The students quickly understood this system of representation and used it to describe configurations. For some of them one could indifferently represent the configuration of figure 1 by (1, 2) or by (2, 1) : the sequence does not matter, it is enough to know that there are 1°) two horizontal lines and 2°) one pawn on one of them and two pawns on the other. 50 to 60% of the students discovered that in fact (1, 2) and (2, 1) are equivalent because they represent the same game. 6 students only, out of 24, were able to use this fact and agreed that writing (1, 2) is more practical than writing (2, 1) because the natural order is respected. From now on we shall call these 6 students, the kernel of the class. The other students either hesitated or rejected the equivalence described above. The discovery of strategies for more complex configurations, by using strategies already known for simple configurations depends heavily on the equivalence described above. It is therefore natural that the students belonging to the kernel were able to carry out a reasoning by induction based on representations, while the others felt the need to go back to concrete situations and were neither able to generalize nor capable of summarizing them.

After 6 weeks (one lesson each week) we had to interrupt this part of our work for 8 weeks. We then resumed our research with a new approach.

Second approach.

In this phase of our work, we submitted some axioms to the children and we wondered whether they would be able to discover some theorems, among which Pick's theorem.

Pick's theorem asserts that the area A of a simple lattice polygon is given by the following formula : $A = I + \frac{B}{2} - 1$, where I and B denote respectively the number of interior and boundary lattice points of the polygon (Liu, 1978).

To introduce those axioms we used a frame, which we called "Nowhereland". The laws of that country are the axioms to investigate. Here are some of them :

Law n°1 : in Nowhereland, people use wires fastened to posts, to mark the boundary of their fields.

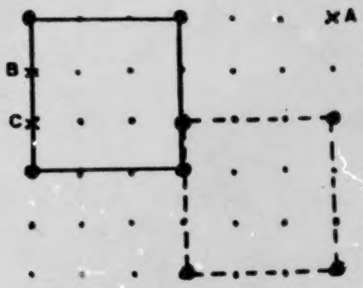
Law n°2 : these posts are placed at regular intervals, the same for the whole country. The unit of length is one "trick". The unit of area is thus ... one "square trick".

Law n°3 : the lines defined by the wires or by their extensions must form right angles. Each right angle has its post.

We used a geoboard to represent fields. At first the students thought that all fields had to be squares. To each field they associated 4 posts only.

Two fields are shown in figure 2. The full line represents the boundary of one field, the dotted line that of an adjacent field. The larger dots represent the first posts accepted by the children and the crosses show the first posts, other than corner ones, identified by our students.

Figure 2.



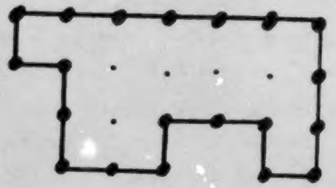
The students observed the fields represented on figure 2. They said that the situation of figure 2 was possible. They then noticed that law n°3 implied the existence of additional posts, such as post 'a' and post 'c'. They finally deduced, from law n°2, the existence of post 'b', etc.

Without hesitation, they suggested that "one trick" should be the interval between two posts. The problem of the diagonals had not been formally solved yet. This was done in the frame of a "Superlaw".

Superlaw : in Nowhereland, to simplify the geometers' work, two perpendicular directions have been chosen once and for all. People call them the vertical and horizontal direction. They have been used to define a squaring of the country. Each side of each square measures one trick. At every intersection of a vertical line and of an horizontal line, a post has been planted. There are no other posts.

This Superlaw is our formulation of the children's discoveries and/or proposals. One of their observations was important : a field does not need to be a square or a rectangle, as long as its sides meet at right angles; but the smallest possible field is the unit square.

Figure 3. : An example of oddly shaped field.



System of representation used.

We suggested the following notations : B represents the number of boundary posts and I the number of interior posts. To each field one associates a couple (B, I). The children quickly found out that fields of different shapes can have the same (B, I), but they noticed that (4, 0) can only represent the unit square. The field of figure 3 is an example of (20, 5). Its area is 14 square tricks.

We used the geoboard to suggest problems : firstly we showed two adjacent fields to the children. To the first one, we associate the couple (B_1, I_1) and to the second one (B_2, I_2). Those two fields have C boundary posts in common. Should both fields become one property (the "union field"), the fence would be rebuilt. What couple should we then associate to that unique field ? Secondly we looked at the field of figure 3. We counted the number of square tricks and noticed that the area of this field is 14 square tricks. We asked the children whether there is any relation between this number and the fact that the field is a (20, 5) !

Results.

The children discovered several different but equivalent formulae. They also discovered and explicitly expressed that equivalent formulae can look very different. They finally found out a counterexample : the formulae are false if $C = 1$.

Here are some examples of formulae proposed for the union field.

1°) $S = (B_1 + B_2) + (I_1 + I_2) - C$.

After experimental checking, a student announced : "S is the total number of posts of the union field, and nothing else".

2°) $B_3 = B_1 + B_2 - 2C + 2$.

The student who suggested the formula proved that it gave the correct number of boundary posts for the union field.

3°) Another student then came to the conclusion that :

$I_3 = S - B_3$.

4°) Another student gave another formula (and a proof) for the number of interior posts of the union field :

$I_3 = I_1 + I_2 + C - 2$.

4 students were able to produce completely formalized proofs ofthe validity of what they called "the union formulae" (formulae 2 and 4). They explained their results to other children and 14 out of 24 eventually understood this. The children proved then that the two last formulae (3 and 4) are equivalent. They were ready to tackle the second problem we suggested : Pick's theorem.

After an interruption of two weeks we resumed the study with one field and its couple (B,I). All the students came to the conclusion that other fields could have the same (B,I). Several students, and this time not only those belonging to the kernel, looked for a method to build such fields. They found several fields by trial and error (but no method). Experimentally they noticed that all those fields had the same area. 50% of the children wished to find a formula enabling them to express directly the area A in function of B and I (i.e. Pick's theorem). The first formula suggested by a student was $A = \frac{1}{4}(B + I)$. This student justified it in the following way : "because a (4, 0) is a square of area equal to 1". Other children found some counter-examples to reject that formula. One week later the students had discovered and formulated in various ways, Pick's formula. They had tested its validity on many fields and concluded that the formula ought to be "universal".

The students tried to prove by induction the validity of the formula they had discovered (Pick's formula). They did not succeed in giving a complete formalized proof. Nevertheless, they discussed how a farmer could buy several fields, unite them in order to have a big field, compute the couple associated to this new field and find out the area of this field. They reached the following conclusion : "If F_1 and F_2 are two adjacent fields, then the 'union formulae' give the couple attached to the union field. If one applies Pick's formula to this new couple, then one gets a number which is precisely the sum of the numbers obtained by applying Pick's formula to the couples attached to F_1 and F_2, i.e. the area of the 'union field'. ".

Their proof lacked two details :

1°) any field is a union of unit squares, and

2°) Pick's formula is valid for the unit square.

We noticed that our group of 24 children consisted in fact of 5 subgroups :

1°) 4 children able to find out and prove new formulae. 3 of them were 11 year old in June 79, the fourth one was 12 years 1 month.

2°) 5 children able to find empirically new formulae but not to formulate explicitly formal proofs : they usually used 3 or 4 examples.

3°) 5 children unable to discover new formulae by themselves, but able to understand them. The 10 children of groups 2 and 3 eventually understood the explanations given by the 4 children of group 1.

4°) 5 children who were only able to apply passively new formulae : these children did not understand what they were doing.

5°) 5 children who were either too lazy, or too stupid, or both, to produce any visible work : it is thus impossible to conclude anything about their logical abilities. 4 out of 5 were older than 12 years and 3 of them were at least one year behind in their school work.

Conclusion.

The children have shown that they are able to discuss in a sensible way and to refer, if necessary, to results previously obtained and they spontaneously looked for them.

The group as such had positive reactions : first of all in formulating ideas and then in controlling them. The friendly understanding prevailing among the children enabled them to discuss with one another, demonstrate the possible mistakes of some of them without hurting their feelings.

As far as the proposed systems of representation are concerned, one must distinguish two phases : during the course of the first one (before the first interruption of our lessons) the students, except those belonging to the kernel, had a tendency to go back very quickly to concrete manipulations. One may wonder whether, for those children, the interruption did not occur at the wrong moment. During the course of a second phase (after resuming the lessons), the system of representation was totally mastered by most children. About 50% of them preferred to manipulate representations rather than concrete situations : they created, explicitly formulated, proved and used formulae, using representations. Their formulae expressed general laws and the use of concrete situations was limited to the checking of particular cases and/or the search for counter-examples.

The research described here is in fact only a clinical observation. It should enable us to prepare a similar program, but intended for younger children who know how to use our formalism. This latter group will be compared with a control group : children of the same age who have not learned to use the formalism introduced to the students of the experimental group.

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ADDITIVE PROBLEMS AND DIRECTED NUMBERS.

Patrick MARTHE.

1. Introduction.

In this text an additive problem is a problem involving additions and subtractions. The purpose of this study is to show the relevance of Gérard VERGNAUD's classification of these problems (1) and the interest of the introduction of directed numbers and their addition by means of some of these problems.

Two directions in this study have been chosen. The first one consists of a test carried out from the first to the fourth forms in French classrooms. The second one is an experiment in two classrooms of the first form and one in one classroom of the second form.

2. The test.

2. 1. The problems.

Problem A.

On Monday afternoon, Peter received 849 francs from James. Then he went shopping and made substantial purchases. On Monday evening, he counted his money and found that he had 475 francs less than in the morning.

How much did he spend on his purchases ?

Problem X.

On Monday afternoon, Peter received 736 francs from James. Then he went shopping and made substantial purchases. On Monday evening, he counted his money and found that he had 239 francs more than in the morning.

How much did he spend on his purchases ?

Problem B.

In the morning, Mr. Dupont had 654 francs in his bank account. He writes a cheque for his purchases in a shop. In the evening, he does his accounts and notices that he now owes his bank 437 francs.

What is the amount of the cheque he wrote for his purchases ?

Problem Y.

In the morning, Mr. Dupont had 752 francs in his bank account. He writes a cheque for his purchases in a shop. In the evening, he does his accounts and notices that he now has 394 francs in his bank account.

What is the amount of the cheque he wrote for his purchases ?

Problem C.

Mr. Dupont owes 783 francs to Mr. Henry. But Mr. Henry also owes money to Mr. Dupont. Taking everything into account, it is Mr. Henry who must give back 248 francs to Mr. Dupont.

What amount did Mr. Henry owe to Mr. Dupont ?

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(1) The theoretical part may be found in [1] and [2] .

Problem Z.

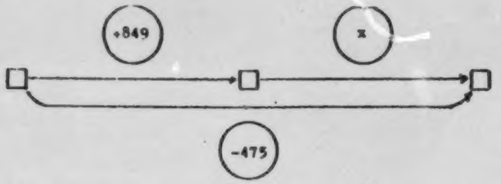
Mr. Dupont owes 684 francs to Mr. Henry. But Mr. Henry also owes money to Mr. Dupont. Taking everything into account, it is Mr. Dupont who must give back 327 francs to Mr. Henry.

What amount did Mr. Henry owe to Mr. Dupont ?

2. 2. The structures.

Structure ttt (problems A and X).

Problem A. (diagram).



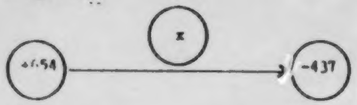
The first square represents the total amount that Peter had in the morning, the second square, the amount that Peter had after James had given him 849 francs and the third square, the amount he had after his purchases. The amounts are states. The children do not need to know them. A question is : "Is the first amount sufficient ?"

"Peter received 849 francs" and "he spent money" are two transformations (t). The composition ot these two transformations is a transformation from the first state to the third state. These transformations are symbolised by circles.

The problem is : "knowing the first transformation and the compound transformation, what is the second transformation ?"

Structure STS (problems B and Y).

Problem B. (diagram).



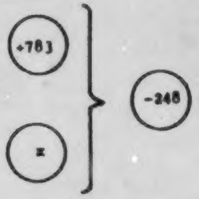
"Mr. Dupont had 654 francs at his disposal in the morning" and "Mr. Dupont owes 437 francs to his bank" are two directed (2) states (S) symbolized by circles. "to write a cheque" is a transformation between two directed states.

The problem is : "knowing the initial state and the final state, what is the transformation between these two states ?"

(2) A state is said to be "directed" when it is associated with a directed number.

Structure SSS (problems C and Z).

Problem C. (diagram).



"Mr. Dupont owes 783 francs to Mr. Henry", Mr. Henry owes money to Mr. Dupont" and "Mr. Henry must give back 248 francs to Mr. Dupont" are directed states.

The problem is : "knowing one state and the resulting state, what is the other state ?"

These problems can be solved by arithmetical means but also by solving the equation $a + x = c$ where a and c are directed numbers. And then, it is possible to consider these contexts as relevant to introduce directed numbers. These problems are in fact solved by the children with arithmetical means. The children know neither the structures ttt, STS and SSS nor their diagrams. Whenever these problems have the same algebraic representation - the equation $a + x = c$ -, they are not solved with equal success by the children and there lies the interest of their study and of Gérard VERGNAUD's categories.

2. 3. The choice of the problems.

All these problems are built from the equation

$$a + x = c .$$

For the problems A (structure ttt), B (structure STS) and C (structure SSS) :

$$a > 0 \quad \text{and} \quad c < 0 \quad \text{and} \quad |a| > |c| .$$

For the problems X (structure ttt), Y (structure STS) and Z (structure SSS) :

$$a > 0 \quad \text{and} \quad c > 0 \quad \text{and} \quad |a| > |c| .$$

All the information given in these problems deals with money. The numbers a and c have three figures. The ordered pairs of numbers (a,c) are different within the six statements A, B, C, X, Y and Z.

As to the choice of the problems among the additive problems :

- firstly, as to the choice of the structures, the results of previous tests on additive problems have been considered, so as to retain problems with which the construction of a commutative group of directed numbers could be easy;
- secondly, as to the choice of the context, the money situation has been retained so as to complete the part in the class dealing with places and journeys.

In some classes booklets of three problems ABC or XYZ (in any order) and in other classes booklets ot two problems AX, BY, CZ, XA, YB and ZC, have been given to pupils. Each pupil has received either a booklet of two problems or a booklet of three.

2. 4. Number and percentage of correct responses.

Structures	problems	1st form	2nd form	3rd form	4th form
ttt	A	1/174 4%	8/174 5%	2/94 2%	2/15 13%
	X	47/79 59%	33/55 60%	41/63 65%	9/15 60%
STS	B	62/175 35%	96/173 55%	69/97 71%	9/15 60%
	Y	80/80 100%	48/54 89%	61/64 95%	15/15 100%
SSS	C	63/168 38%	93/173 54%	63/95 66%	10/15 67%
	Z	58/73 79%	37/54 69%	51/62 82%	13/15 87%

2. 5. Conclusion.

When $|a| > |c|$, a and c having opposite signs, a problem from the structure ttt is actually more difficult than a problem from the structures SSS and STS, these two problems being equally difficult.

When $|a| > |c|$, a and c having the same sign, a problem from the structure ttt is more difficult than a problem from the structure SSS, itself being more difficult than a problem from the structure STS.

For the structure ttt, a problem where a and c have opposite signs is actually more difficult than a problem where a and c have the same sign.

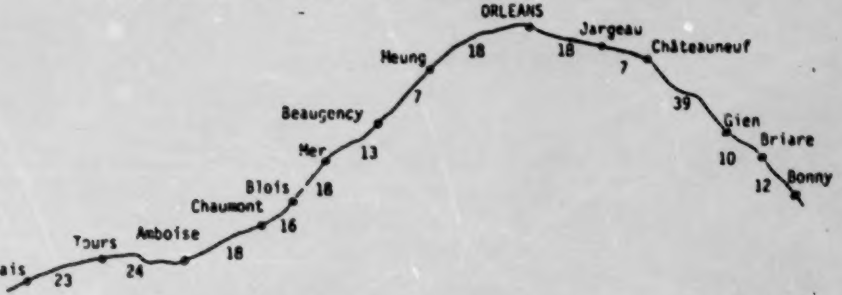
For the structures STS and SSS, a problem where a and c have opposite signs is more difficult than a problem where a and c have the same sign.

In general, correct responses increase with the forms.

3. Experiments in classrooms.

Experiments were conducted in the first and second quarters of the year 1978. Problems concerning journeys, starting-points and destinations on a road along the "Loire" near Orléans, where the pupils live, were proposed.

The pupils had the following map :



The numbers represent the distances (in kilometers) between two towns.

The pupils had to write beside a straight line the distances between Orléans and some other town.

Then they were given two problems :

"A car starts off from a town 18 km from Orléans and travels 56 km.

In which town, or near which town, does it arrive ?"

"A car starts off from a town 25 km from Orléans and travels 7 km.

In which town, or near which town, does it arrive ?"

Each of these problems have four solutions. The pupils did not consider all the solutions. Their solutions were collated. Then the teacher said to the pupils :

"You must distinguish between a place above Orléans and a place below Orléans, the two places being at the same distance from Orléans. You must distinguish between a journey made downstream and a journey made upstream, at equal distances. Give signs to write before the numbers measuring the distances so as to realise these distinctions."

The signs retained were :

v for a place downstream (in French : aval),
m for a place upstream (in French : amont),
↓ for a journey made downstream,
↑ for a journey made upstream.

Then the course consisted in solving problems related to the structures STS and TTT and the equations $a + b = x$, $a + x = c$ and $x + b = c$.

Here, a place is a directed state (S) and a journey is a transformation (T).

4. The test following the course.

4. 1. The problems.

S_1 : initial state ; S_f : final state ; T_3 : compound transformation.

Problem 1. (The car moves off from the zero point;
structure S_1TS_f , S_f unknown)

A car starts off from the place marked (v35). It travels 20 km downstream. What is the directed number of the place where it arrives ?

Problem 2. (The car moves nearer to the zero point;
structure S_1TS_f , S_f unknown)

A car starts off from the place marked (v40). It travels 5 km upstream. What is the directed number of the place where it arrives ?

Problem 3. (The car goes through the zero point;
structure S_1TS_f , S_f unknown)

A car starts off from the place marked (v20). It travels 25 km downstream. What is the directed number of the place where it arrives ?

Problem 4. (structure S_1TS_f , T unknown)

A car starts off from the place marked (v5) and arrives at the place marked (v40). Describe the journey it made.
(journey : direction and number of kilometers)

Problem 6. (structure S_1TS_f , S_1 unknown)

A car travels 30 km downstream. It has arrived at the place marked (m20). What is the directed number of the place from which it has started ?

Problem 3. (structure $T_1T_2T_3$, T_3 unknown;

T_1 and T_2 have opposite signs ; $|T_1|>|T_2|$)

A car makes an initial journey of 30 km downstream. Then it makes a second journey of 20 km upstream. What is the journey which brings it straight from its starting-point to its destination ?

Problem 7. (structure $T_1T_2T_3$, T_2 unknown;

T_1 and T_3 have the same sign ; $|T_1|<|T_3|$)

A car makes an initial journey of 40 km upstream. Then it makes a second journey. If it had made only one journey straight from its starting-point to its destination, it would have made a journey of 60 km upstream. Describe the second journey.

Problem 8. (structure $T_1T_2T_3$, T_2 unknown;

T_1 and T_3 have the same sign ; $|T_1|>|T_3|$)

A car makes an initial journey of 55 km upstream. Then it makes a second journey. If it had made only one journey straight from its starting-point to its destination, it would have made a journey of 30 km upstream. Describe the second journey.

Problem 9. (structure $T_1T_2T_3$, T_2 unknown;

T_1 and T_3 have opposite signs; $|T_1|<|T_3|$)

A car makes an initial journey of 20 km upstream. Then it makes a second journey. If it had made only one journey straight from its starting-point to its destination, it would have made a journey of 25 km downstream. Describe the second journey.

4. 2. Results.

A problem was considered as "successfully solved" if :
- the right absolute value was given,
- the signe of the directed number was the right sign or the sign v (resp. m, ↓, ↑) was written instead of the sign ↓ (resp. ↑ , v, m); because, for instance, a pupil may indicate by the same sign the fact of making a journey downstream and the fact of being downstream,
- the error of calculation consisted in not considering the carry over.

Percentage of correct responses.

structure	STS	STS	STS	STS	STS	TTT	TTT	TTT	TTT	n
unknown	S_f	S_f	S_f	T	S_I	T_3	T_2	T_2	T_2	
problem	1	2	5	4	6	3	7	8	9	
1st form $_1$	61%	17%	17%	43%	48%	30%	30%	9%	13%	23
1st form $_2$	59%	41%	50%	50%	77%	32%	41%	36%	18%	22
2nd form	82%	50%	50%	59%	50%	64%	50%	41%	32%	22

n : number of pupils

4. 3. Commentaries.

For the structure STS, when the final state is unknown, a problem where the car moves off from the zero point is easier than problems where the car moves nearer to the zero point or goes through the zero point.

In general, a problem from the structure TTT is more difficult than a problem from the structure STS.

When two transformations have opposite signs, the problems from the structure TTT appear very difficult for children, even for those who are in the fourth form. In the development of knowledge, this structure is an epistemological obstacle, whereas it is not the case for the other structures.

5. General conclusion.

Two directions of work had been chosen : the first was to carry out a test on some problems, the second to conduct an experiment in the classroom, followed by another test. The first can be carefully controlled. The second is less easy to determine. For the second, the matter is not so much to separate the wheat from the chaff, as to use the observations (whatever the criticisms against them) to improve upon previous hypotheses, to find new ones, or for lack of these to pose new questions and also to prepare future courses and to direct the selection of subjects for discussion in these courses.

The analysis of the results of a test can give indications of the level of difficulties of problems and enable us to compare some with others. Although it can contribute to the choice of subjects in a course, in no case does it provide a substitute for the course itself. In a class, if the goal is not to obtain the repetition of algorithms by children, but if it is established as the acquisition and development of genuine knowledge, the children must be presented with problems rich in content, so that they can assimilate the concepts.

These two directions of work are not contrary; they could be complemented by other ones. The statements and conclusions which can be given must not be intended as a lesson for the teacher, but must play a part in his thought concerning classroom work.

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In the same area, other ideas may be found in papers of A. Bell, J. Brun, M. Brown, F. Conne, T. P. Carpenter, J. M. Moser, P. Katriel, P. Nesher, E. Teubal, D. Küchemann, K. Hart, G. Vergnaud, ...

Problem Solving with the ONE WAY PRINCIPLE

by
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1. The One Way Principle

To understand functions $x \xrightarrow{\textcircled{f}} y$, including in particular the four basic operations of addition, subtraction, multiplication and division, means to comprehend (x,f,y) as a unity. We suggest that the three different problems

(a) Given x and f, find y
(b) Given y and f, find x
(c) Given x and y, find f,

are all approached using the same algorithm. We shall call this the One Way Principle (OWP). The first step is for the student to learn the algorithm for the "direct" problem (a). He then uses this same algorithm combined with guessing and testing for problems of the type (b) or (c).

For example, suppose we had a problem of type (b). An arbitrary choice of x leads to a "wrong result" for y. But the magnetude of the mistake gives hints for a better choice of x. This next choice of x gives a better "wrong result" for y, and gives new hints for a further choice. Step by step the student builds up an intuitive model about the relations and rules between x,f and y. He builds up a model of the function $x \xrightarrow{\textcircled{f}} y$ in an empiric way, like a scientist.

2. Examples

We tested the OWP in some case studies. The students solved the problems either by mental arithmetic or with a simple hand-held calculator (HHC). Up to now we cannot report on long term effects.

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2.1. Mental Arithmetic. Observing children we found that many of them solved $4 \cdot \square = 24$ or $\square + 9 = 15$ by guessing and testing and not by applying the inverse operations.

2.2. Addition and Subtraction. Two students with one calculator play the game "Big Zero". One of them stores $\textcircled{-a}$ and the other one tries to find "a" by searching the input x getting the output 0 in $x \xrightarrow{\textcircled{-a}} 0$. A wrong input yields a nonzero output, but it gives hints to a better choice of the next input. The game trains a feeling for the order of magnitude of numbers.

2.3. Multiplication and Division. As in 2.2 two students with one calculator play the game "Big One": $x \xrightarrow{\textcircled{:a}} 1$. The game trains an intuitive feeling for ratios and proportions. Two other games have similar goals. In "Hit-the-Target $x \xrightarrow{\textcircled{\cdot a}} z$" we know a and the target $z \in \mathbb{N}$. We have to find a number x with $\mathrm{INT}(x \cdot a) = z$. Usually x will be a decimal number. In "FF" (Factor Finder) we also work on the problem $x \xrightarrow{\textcircled{\cdot a}} z$. But here we know a first input x_1 and the target $z \in \mathbb{N}$ and are supposed to find a_1 with $\mathrm{INT}(x_1 \cdot a_1) = z$. Having no success in the first trial our next input will be $x_2 = x_1 \cdot a_1$ and we have to guess a_2 with $\mathrm{INT}(x_2 \cdot a_2) = z$ etc. All x_i form a sequence with limit z and all a_i form a sequence with limit 1. Hit-the-Target and FF become easier by taking an interval for the target instead of a single number z.

2.4. Ratio and Proportion. This understanding is reinforced by the use of real life situations involving measuring, weighing machines, gas-station, speed, exchange... Guessing and testing to fill in incomplete information in such situations is done via the calculator, estimation or mental arithmetic.

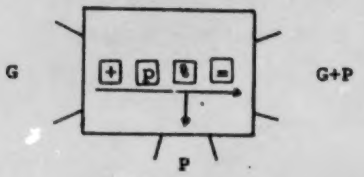
2.5. Percentages. We reduce all problems concerning the two

formulas $\frac{p}{100} = \frac{P}{G}$ and $\frac{G \pm P}{G} = \frac{100 \pm p}{100}$

to the one percentage function which is given on certain calculators in the following way:

$\boxed{6}\,\boxed{3}\,\boxed{5}\,\boxed{+}\,\boxed{6}\,\boxed{\%}\,\boxed{=}$ (673.1).

This sequence of keys corresponds completely to our every day language and therefore is easy to remember. Starting with the function $G \xrightarrow{\boxed{+}\,\boxed{p}\,\boxed{\%}\,\boxed{=}} G+P$ and applying the OWP we develop a percentage feeling by guessing G or p and testing with the calculator. Immediately after pushing the $\boxed{\%}$ -key we see in between the output P which enables us to widen the concept of the $\boxed{\%}$ -key:



Next we complete our (intuitive) percentage feeling with the OWP-training of $G \xrightarrow{\boxed{+}\,\boxed{p}\,\boxed{\%}} P$ which is - strange to say - identical to $G \xrightarrow{\boxed{-}\,\boxed{p}\,\boxed{\%}} P$. Remembering 2.3 and 2.4 we feel proportion and we find

$$G \xrightarrow{\textcircled{\cdot \frac{p}{100}}} P \quad \text{and} \quad G \xrightarrow{\textcircled{\cdot \frac{100+p}{100}}} G+P.$$

Using the spiral approach we now discuss percentages without the $\boxed{\%}$ -key.

2.6. Square roots. To find the side of a square with given area we guess and test by squaring. Trial and error trains a feeling for the order of magnitude of numbers, the continuity of the number line, the monotony of the function, for nests of intervals and for convergency.

3. Summary

The OWP helps to develop an operational understanding of functions. We start with the black-box algorithm. Then by trial and error we develop an intuitive feeling for the range of the algorithm. When this feeling has become conscious the algorithm can be described algebraically. Now we ask for the inversion of the function, its algebraic description, for formulas, and for algebraic solutions of the problems (a) - (c).

4. Open Questions

We start with the drill of an algorithm. The understanding of the algorithm and the knowledge about the range of the algorithm follow afterwards. This leads to some open questions:

4.1. Is it allowed to train children in activities which they do not understand (at least in the beginning)? Where do children learn in a similar way? Where do we accept this kind of learning? Where do we reject this procedure as a teaching method and why?

4.2. How successful is the learning of intellectual concepts via feed back by trial and error? Are there results of research which prove or tend to prove that solving the problems (b) and (c) by a trial and error procedure occurs only randomly without any aspects of feed back, while under the same conditions other teaching methods lead to an operational understanding of (x,f,y)?

4.3. Which is the long term success in memorizing and applying a black-box algorithm, when the training of this algorithm only took place via drill-and-practice without developing an unconscious or conscious understanding? Where is research to that question concerning the learning of mathematical concepts? Where is a similar research to that question concerning the learning of other concepts?

4.4 There are two teaching methods. Either we start with a black-box mechanism (algorithms for the solution of a class of problems) and develop understanding afterwards via the OWP or we first develop understanding for the problems (e.g. by examples and by explaining and discussing) and try to train the algorithms afterwards which are necessary to solve the problems of that class. Which is the long term difference between these two methods in regard of memorizing, applying and understanding of the algorithms? Where is research to that question concerning the learning of mathematical concepts or concerning the learning of other concepts?

4.5. How do the answers in 4.1 to 4.4 depend on the type and the complexity of the black-box mechanisms (to work with concrete materials, to process a sequence of instructions, to run a sequence of stimulus-response reactions, to apply a sequence of calculator key strokes, to select and to use a formula,...)?

Applying Mathematical Generalization in the Acquisition of Conceptual Structures in Trigonometry

Harold Mick and Gerald Brazier

There has been considerable controversy over the most appropriate mathematical setting for introducing the trigonometric functions. Proponents of the right triangle interpretation argue that concrete experiences give students a feeling of reality and practical applications on which to build future concepts. Opponents argue that right triangle trigonometry is out dated, and instead the trigonometric functions should be introduced as functions of real numbers to provide appropriate experiences for the study of calculus.

Research has not been that helpful in resolving the controversy, though Strong (1975) reported significant differences of an angular approach over a wrapping function approach after 15 weeks of treatment ($p < .05$) and two weeks later on a retention test ($p < .01$); however student interviews revealed considerable confusion and lack of mastery of basic concepts for many students.

Textbooks reflect these conceptual differences by distinguishing between circular and trigonometric functions -- depending on whether the conceptual bases reflect the unit circle on the one hand or coordinates in the Cartesian plane on the other. Circular functions are typically presented first, followed by trigonometric functions, and finally right triangle applications are discussed. Ideally it would be desirable if students acquired all these interpretations of the trigonometric functions and understood their interrelationships, however there is evidence (Mick and Brazier, Note 2) to support the position that students consider each interpretation to be associated with a completely separate problem.

These distinctions among interpretations are important for the psychology of learning mathematical concepts. Conceptual structures become more sophisticated and powerful with increased integration of existing concepts; whereas mental structures consisting of bits and pieces give the learner a psuedo-security which is frequently revealed when he is presented a novel problem. The resolution of the controversy, then, does not appear to depend so much on whether one interpretation is better than another; but rather, how can the desired integration of all interpretaions be best accomplished?

A theoretical answer to this question may lie in the domain of the psychology of learning mathematics, where abstraction and generalization processes explain the acquisition and integration of mathematical concepts. It is the major purpose of this paper to illustrate the application of cognitive processes associated with mathematical generalization as they apply toward the acquisition and integration of trigonometric concepts. First, an account of mathematical generalization will be presented; and then, the associated cognitive processes will be applied in the context of generalizing from first quadrant angles and triangles to angles whose terminal sides lie outside the first quadrant.

Mathematical Generalization

Dienes (1961) defines mathematical generalization as follows:

> A class B is a mathematical generalization of the class A if B includes as part an isomorphic image of A, in relation to all relevant properties. This means that the classes A and B could consist of quite different elements, as long as there was part of B which was somehow an exact image of A, mirroring the properties of A in all relevant respects. (p. 282)

For example, let the whole numbers be designated by class A. Then the fractional numbers, denoted by class B, is a mathematical generalization of the whole numbers since B includes as part an isomorphic image of A, namely, those fractions in the form 0/1, 1/1, 2/1, 3/1, etc., with respect to the usual addition and multiplication.

Skemp (1971) has described the mental processes involved with mathematical generalization in three phases:

> 1. From a set of examples, a general method is derived,
>
>   which can be applied to other examples of the same kind.
>
> 2. The method is next formulated explicitly, considered as an entity in itself, and its structure analyzed.
>
> 3. This structure is used to invent ways of using the same method for examples of a new kind. The original examples are included in the enlarged field of application of the method. (pp. 60-61)

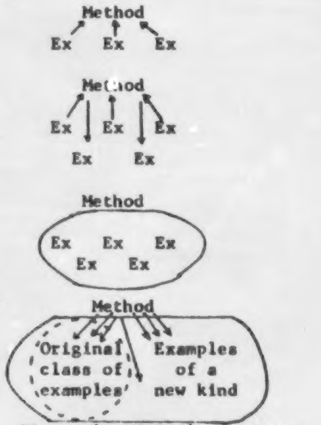


Figure 1. Mental processes of mathematical generalization.

It appears that both Dienes and Skemp are discussing the same concept; however, they differ somewhat in their delineation of the steps leading to the culmination of the process. While Skemp (1971) describes a linearly ordered sequence of three phases, Dienes (1961) lists four requirements (not necessarily in any order of acquisition):

> (1) Abstraction processes must be completed so that the structure of the individual elements of all the classes involved is realized.
>
> (2) The three classes A, A' and B must be well formed.
>
> (3) The isomorphism between A and A' must be established.
>
> (4) The inclusion of A' in B must be realized.
>
> When all the above mental processes have been completed, it is still necessary to integrate them into the structure in the figure (see Figure 2). It is only when this integration has taken place that the mathematical generalization has been completed. (p. 292)

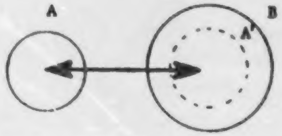


Figure 2. Class structures of mathematical generalizations.

Synthesizing the work of Skemp and Dienes has led to the following sequence of events that describe the psychological processes emboding mathematical generalization in such a manner that they can hopefully be used as guidelines for curriculum development (see the next section of this paper). The first and second events provide preparation for the mathematical generalization processes that are then described in events (3) - (5).

Event (1). Skemp's first phase is the same as Dienes' (1961) abstraction process: "The process of abstraction is defined as the process of drawing from a number of different situations something which is common to them all" (p. 281). This process is referred to by many authors as concept formation.

Event (2). Skemp's second phase corresponds closely to Dienes' (1961) formulation of primitive generalization:

> Primitive generalization is the passing from one class to another where the other includes the former as part. (p. 282)
>
> ....
>
> The third stage in the cycle[1] will have to contain a modicum of logical realization, although the implicit type of thinker will still seek this in practical situations which will mirror the logical structure perceived at the moment of insight. (p. 289)
>
> ....
>
> The abstraction necessary to recognize the eligibility of the elements for the formulation of classes, and the generalization of realizing just how extensive these elements are, might happen at almost the same time and so appear as one process. (p. 290)

Perhaps the most important aspect of this step for curriculum development is to provide sufficient experiences to ensure that the learners detach the operation from a dependence on particular examples. This detachment of form from content is needed preparation for the mathematical generalization processes that formally begins with Event (3).

Event (3). The acquired structure resulting from the first and second events is centered around a particular method or operation (unitary, binary, etc.). It is the desire and practicality to generalize this operation that is used to create new objects on which to operate. This cognitive process of creating new objects is called extending a set by Laenen (Note 1; also see Laenen's paper in these proceedings); and the process is accompanied by an awareness of what is psychologically and mathematically taking place -- the teacher being almost indispensable in seeing that learner awareness does happen at the appropriate times.

Event (4). The learner is now ready to generalize the operation over this new and enlarged set of objects. From a psychological point of view the learner needs to exercise care to generalize the operation over the newly created objects so that the "old" structure and governing principles are preserved. Of course some new principles may develop during the generalization process.

Event (5). This last event serves a putting-it-all together function -- somewhat in the spirit of Dienes' description (see Figure 2). In brief: properties of the operation and associated theorems may be generalized; the isomorphism is emphasized and the operation is applied to the isomorphic images of the old objects; new names and symbols are invented as necessary; and reflective activities are used to assist the learner to consolidate the entire structure in the logical perspectives of proper isomorphisms and inclusions.

Application

The process of mathematical generalization suggests that the acquisition of the trigonometric functions begin in their most rudimentary form (from angle measure) and then progress through successive abstractions and generalizations. Using the sine function for illustrative purposes, the first event outlines the abstraction of sine as an unitary operation: (1) given an angle measure less than a right angle, say in degrees, (2) form a right triangle for which one of the acute angles possesses the given angle measure, then (3) form a ratio of the length of the side opposite the angle of given measure to the length of the hypotenuse, and finally (4) the sine of the angle measure is defined to be the value of the ratio found in step (3).

As the operation is gradually acquired (the learner can successfully apply the operation to sample angle measures), the inverse operation should also be abstracted: (1*) given a ratio (a real number between zero and one), (2*) form a right triangle such that the lengths of the side opposite one of the acute angles to the hypothenuse form the given ratio, the (3*) measure the respective acute angle, say in degrees, and finally, (4*) the measure in degrees found in step (3*) consitutes the inverse sine of the given value between zero and one.

The newly acquired operation needs considerable practice in a variety of situations in order to become a viable part of the learner's conceptual structure. A premature instructional move to the second event can wip out tentative gains made during the first event. For example, considerable time should be spent on realizing that sine is a ratio and thus can be found in a class of similar triangles for any angle of given measure between 0 and 90 degrees. Also, finding missing parts of right triangles provides needed practice in a variety of situations.

Gradually the learner is presented all six trigonometric functions in both forward and inverse forms. Relationships between functions are noted. For example, $\sin(90 - x) = \cos(x)$, $\sin^2(x) + \cos^2(x) = 1$, $\tan^2(x) + 1 = \sec^2(x)$, $\tan(x) = \sin(x)/\cos(x)$, and eventually the sum and difference formulas are acquired. As previously noted, the second event is complete only when the essential features of the operation become independent of the examples from which it was derived. Essentially a mini-trig course will be developed around the right triangle interpretation. Identities will be derived and relationships with other functions will be secured.

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[1] Dienes draws from Piaget's sensory-motor, concrete operational, and formal operational stages of development to construct a paralleling three-stage cycle for the formation of abstractions and primitve generalizations.

The third and fourth events are very complementary in this particular trigonometric application, partly because angles with measures outside the zero to ninety degree interval are rather common place and known to most students, although the sine of the measures of these "new" angles are not generally known to these same students. So the third and fourth events begin with a practical situation, that by its very nature fosters an urge to venture forth and invent "new examples". An appropriate situation for creating this conducive environment is the group of sum and difference identities. For example,

$$\begin{aligned} \sin(120^\circ) &= \sin(60^\circ + 60^\circ) \\ &= \sin(60^\circ)\cos(60^\circ) + \sin(60^\circ)\cos(60^\circ) \\ &= \sqrt{3}/2 \cdot 1/2 + \sqrt{3}/2 \cdot 1/2 \\ &= \sqrt{3}/2 \end{aligned}$$

And similarly $\sin(-30^\circ)$ can be shown to equal $-1/2$ if the sum and difference formulas are to hold for these new angle measures.

Although $\sin(120^\circ)$ and $\sin(-30^\circ)$ make no sense in the original structures of right-triangle trigonometry, it is possible to "illegally" apply existing relationships, specifically the sum and difference formulas in this case, to these new situations and arrive at results consistent with the old structures.[2] Further applications should be experienced in order to see developing patterns and structures.

But in what sense has the original structure been subsumed? What is the context or conceptual base of the enlarged class of examples? What do the old examples look like in the new setting? These questions signify the beginning of the fifth event, and initiate the search for an appropriate interpretation such as, for example, the Cartesian plane in which measures of all angles can be interpreted in a manner facilitating the generalization, but at the same time preserving the old structure. In effect, there is an unique class of similar right triangles associated with each given angle (see Figure 3).

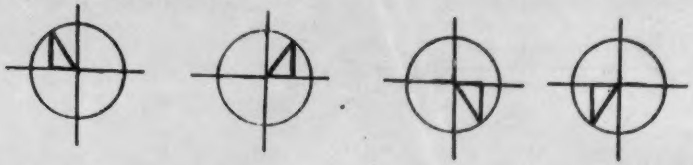

Figure 3. Generalizing the right triangle interpretation.

Of course much practice is needed to reinforce the newly extended operations, but equally important is the need for activities requiring the learner to reflect on the generalization -- relationships involving both the isomorphism and inclusion. It is crucial that the overall structure is internalized as an integrated whole. Indeed, it is the position of this paper that the acquisition of the total integrated structure can best be realized through the psychological processes embodied in mathematical generalization. In other words, the controversy between right triangle and unit circle interpretations has been answered -- start with angle measure in right triangles and then generalize to angle measure outside the zero to ninety interval by utilizing mathematical generalization processes.

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HIERARCHY AND PROCESS IN THE CONSTRUCTION OF THE GEOMETRICAL FIGURE IN THE CHILD AND ADOLESCENT[1]

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INTRODUCTION

Two basic concepts epitomize the change developmental psychology is bringing to psychology and education: the concepts of *hierarchy* and *adaptive differentiation:*
i) Development proceeds hierarchically. Each new structure integrates the preceding one as a particular substructure. These structures form *stages*.
ii) The change from one structure to the next proceeds through *differentiation*. Schemes are structured behavior, which solve particular problems. Now problems in the outer world can become more complex. A scheme, which can no longer cope with some new aspect of a problem, undergoes an internal change. This allows the scheme to grasp the new aspect of the problem and relate it to the aspect already known. This is a type of adaptation of a subject to reality which Piaget (1975,1977) calls *"increasing equilibration"*. We understand it as a process of differentiation or *adaptive restructuring* of the scheme, enabling it to integrate the new aspect of a problem. This concept of development will be illustrated here by two experiments: *"Drawing geometrical figures"* and *"Mechanical curves"*. First results of the first experiment were given by Noelting (1974).

METHOD

EXPERIMENT 1: DRAWING GEOMETRICAL FIGURES

Instrument: A series of 16 geometrical figures has been made up to study development between symbolic thought (2 years) to higher concrete operations (10-11 years). The simpler ones had already been studied by Gesell (circle, plus, square, X, triangle, rhombus). To these were added some figures used by Piaget and Inhelder 1947, 1956 (disjoint, tangent, intersecting circles, snowshoe) and Rey (box shown in perspective, staircase). Some new figures were devised to study the operational aspects of figure construction (inscribed rhombus, flag, shifted squares, parallelogram with segments). (See Figure 1.)

Sample: First an experiment was undertaken with twenty (20) children between 4 and 9 years of age, in order to determine the order of difficulty of the figures. A definite sample was then chosen with sixteen (16) children per age from ages 2 to 11. The subjects were chosen at random in four districts of a town in Quebec, each corresponding to a particular socio-economic level, with two (2) boys and two (2) girls in each district. Children from ages 5 up were chosen at random in a school in each district, children younger were selected at random in the same district and interviewed at home.

Procedure: The figures were presented successively to each child. He was asked to copy the figure as best he could, and in case of failure, given two more tries, asked to "try better".

Correction: Criteria for accepting a figure as correct were set, with examples of success and failure. Correction was dichotomic and a further item added: scribble, for children unable to reproduce correctly even the easiest figure. Thus 17 items were obtained with one hundred and sixty (160) subjects.

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[1] This research was supported by FCAC grants from the Ministry of Education of Quebec.

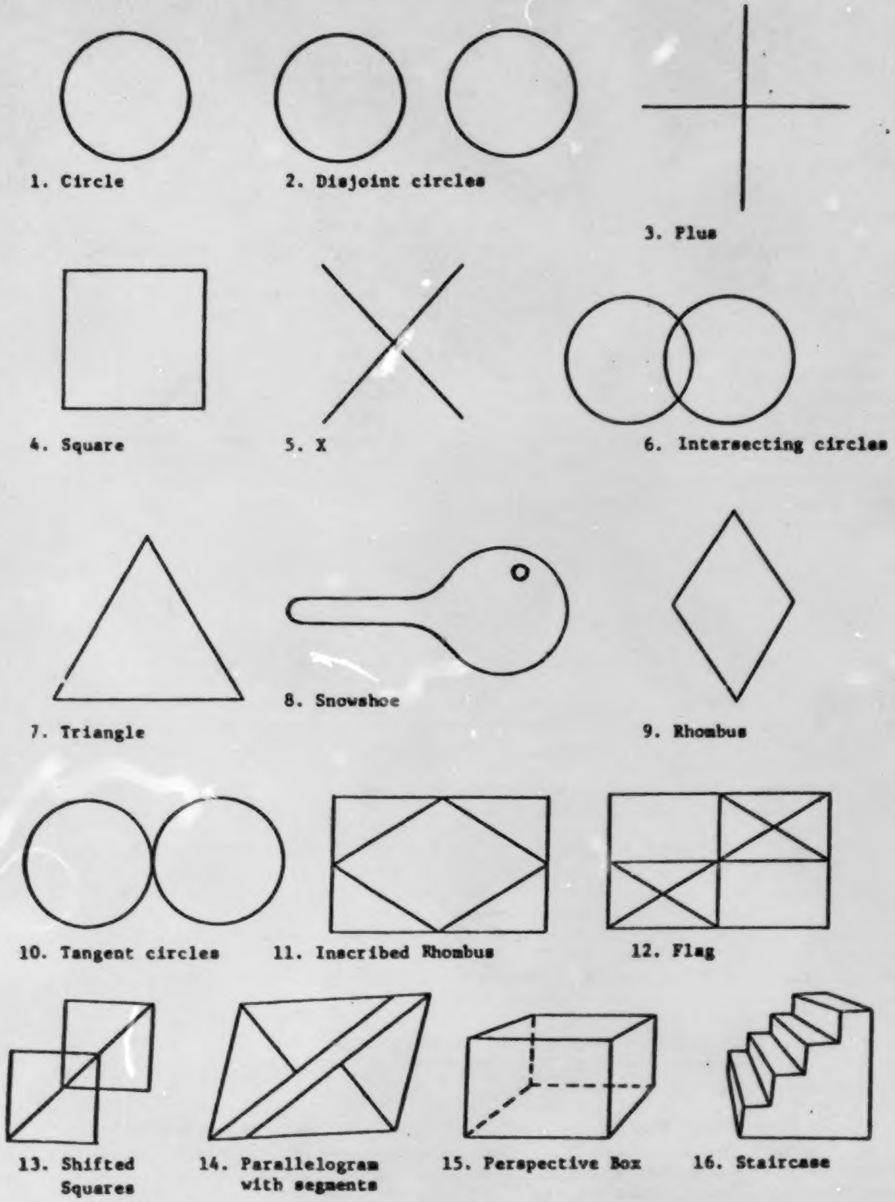

FIG. 1 The sixteen forms of Drawing geometrical figures.

Treatment of results: After correction, results underwent treatment at two levels:

i) A *first-order analysis* was made to determine *stages*, i.e. structures in behavior hierarchically integrated. This consisted of a scalogram analysis, categorization of items, regrouping of subjects succeeding at items of each category, and chronological differentiation.

ii) A *second-order analysis* then followed to determine a *pattern* in the succession of stages, i.e. a structure of structures. Comparing erroneous and successful strategies with item structure at each successive level led to the concept of *adaptive restructuring*, i.e. change in structure to fit new data in the environment.

Piaget (1975, 1977) had introduced the concept of *increasing equilibration* to explain process in development. Noelting (1978) had refined this model to account for the stages found in the development of the concept of ratio in the child and adolescent. He had determined four *phases* in a period of equilibration, when a scheme is confronted with a new aspect in a problem. These were the α-*phase* or *centration* on the familiar aspect of the problem, β-*phase* or *dualism* between familiar and new aspects, ρ-*phase* or *relation* between these aspects, and γ-*phase* or *hierarchical integration* of these two aspects in a new structured whole. Certain problems are structured through a number of *periods*.

These two orders of analysis were applied here.

EXPERIMENT 2: MECHANICAL CURVES

In order to study how geometrical concepts develop at the formal stage of thinking, the experiment bearing on *Mechanical curves* (Piaget, Inhelder and Szeminska, 1948, 1960) was studied with a simpler instrument.

Instrument: A board surrounded by a ½cm frame is placed in front of the subject. On the board is a sheet of white paper. Three objects are given in succession, a square 8 x 8 cm, an equilateral triangle having a side of 8cm, and a circle of diameter 8cm. The subject is asked to roll the object slowly from the left hand corner to the right, resting it continuously on the base and without sliding it. The subject is made to imagine a pencil at one vertex of the square or triangle or one point on the circle's circumference. He is asked to think out what figure would be drawn on paper when the object rolls, draw it, and explain how he found out.

Sample and Procedure: One hundred (100) subjects from five age levels between 8 years and 24 were given the experiment in an individual setting. A "clinical method" of questioning was used and protocols written out with drawings.

Correction: Answers which seemed similar were differentiated as categories, and subjects distributed in each.

Treatment of results:

i) A *first-order analysis* was made to determine the particular strategy put to use for each category. Chronological comparisons were made of age distribution of subjects of each category. Significant differences and structural hierarchy lead to defining *stages*. Then errors at each stage were analyzed in terms of confusion between relevant aspects of the problem. Strategies characteristic of each stage were described.

ii) A *second-order analysis* lead to interpret particular stages, as *phases* in a sequential development, where certain aspects of a problem are seized, others neglected. A comparison of the two experiments led to integration of their levels. This treatment is not yet terminated.

RESULTS

1. *First-order analysis*

Experiment 1.

Results were submitted to a *scalogram* analysis with a BMD 05S program, Guttman Scale No. 1 (Dixon, 1971). A CR of .97 was obtained, with MMR = .75 and PPR = .88 . This makes the scale a so-called perfect hierarchy. *Categorization* was introduced to separate items into stages. Subjects succeeding at an item of a stage, and missing all items of the next stage, were grouped. Results are given in Table 1. *Chronological differentiation* was undertaken by comparing the age distribution with the Kolmogorov-Smirnov Test. The high coefficients obtained at the Guttman Scale can be explained by dichotomic correction of items and strict unity in content.

Experiment 2.

Experiment 2 is still in an exploratory phase. Four stages seem to come up, which are chronologically differentiated. The last corresponds to Higher Formal Operations.

2. *Structural description of stages and second-order analysis*

Behavior at the successive stages of development will be described, with a tentative interpretation (given in brackets) of processes involved in the passage from a stage to the next. This should not be taken into account at the first reading of the data obtained.

PERIOD I: SYMBOLIC PERIOD (1-2 to 3-4 years).
From scribble to square: Adapting movement to cue, with gradual mastery of closure and vertical-horizontal directions.

Stage 0: Sensori-motor (1-2 years). Pure scribble.

In scribble, two distinct rhythmic movements are found: a circular movement enclosing a portion of space, and a to and fro movement, not quite straight, but made in a certain direction (Fig. 2). At first the child does not vary his movements in function of the figure presented. There is thus *centration* on the movement itself, as distinct from cues in the outer world. (This can be considered an α-*phase* of equilibration, where the subject simple *ignores* the disturbance in the environment.)

Stage IA: Lower Symbolic (1-2 to 2-3 years). Varied scribble, with final success at circle and line segment.

Gradually scribble is varied according to stimulus. When the circular movement is interrupted at its beginning, the circle is mastered. When the to and fro movement is made in one direction only, a threadlike line up obtained, which is a precursor of the line segment, with its property of collinearity. Two distinct figures

TABLE 1

FREQUENCY OF SUCCESS ACCORDING TO AGE AT FORMS OF PROGRESSIVE FIGURES FORM B,
WITH SPATIAL CATEGORIZATION OF FORMS AND CHRONOLOGICAL DIFFERENTIATION TO CONSTITUTE STAGES

Age	N	Item and stage																	
		O	IA		IB		IIA			IIB		IIC		IIIA		IIIB			IVA
		pure scribble	varied scrib-ble	○|	○○	+	□	×	⊕	△	⚲	◇	8	⊠	▦	⧉	▨	⊞	⌂
2	16	2	7	7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	16	1	4	6	3	2	-	-	-	-	-	-	-	-	-	-		-	-
4	16	-	-	3	1	2	2	2	3	1	1	1	-	-	-	-	-	-	-
5	16	-	-	-	-	1	1	-	2	2	1	4	4	1	-	-	-	-	-
6	16	-	-	-	-	-	-	-	-	2	2	5	3	3	1	-	-	-	-
7	16	-	-	-	-	-	-	-	-	1	-	4	1	2	2	3	3	-	-
8	16	-	-	-	-	-	-	-	-	-	-	3	2	2	1	3	2	3	-
9	16	-	-	-	-	-	-	-	-	-	-	2	2	1	6	-	2	2	1
10	16	-	-	-	-	-	-	-	-	-	-	1	-	1	3	2	2	4	3
11	16	-	-	-	-	-	-	-	-	-	-	-	-	2	3	3	5	1	2
Total	160	3	11	16	4	5	3	2	5	6	4	20	12	12	16	11	14	10	6
p^a		-	-		<.03		<.03			<.05		<.07		<.01		<.15			<.10

Note. - [a] Level of probability of difference in the age distribution of two adjacent stages (Kolmogorov-Smirnov Test).

are drawn on paper, obtained through coordination of movement with cues. This is illustrated in Figure 3. (This behavior seems related to a β-*phase*, with *categorial differentiation* or *dualism.*)

Stage IB: Upper Symbolic (2-3 to 3-4 years). Mastery of disjoint circles and plus.

Success at the disjoint circles is interpreted as a relation established between two geometrical figures through *separation* in space, while formerly these circles were drawn one atop or inside the other. Likewise, success at the vertical-horizontal cross is a relation between two line segments in *reciprocal directions*, while formerly these lines were loosely scattered on paper. This is illustrated in Figure 4. (This behavior seems typical of a ρ-*phase*, or *relational differentiation* betweem entities.)

Stage IIA: Lower Intuitive (3-4 to 4-5 years). Mastery of the square and intersecting circles.

Mastery of the square involves coordination of distance and direction, vertex and closure. (Coordination of length and direction, vertex and closure would make this a γ-*phase* of *hierarchic integration*, corresponding to the achievement of a first period of development.)

Mastery of the intersecting circles, in a similar way, involves coordination of both separation and union. Rectilinear and curvilinear figures seem to develop in a parallel way all along Period I. Lucito 4;0, in Figure 5, is an illustration of the mastery of rectilinear vertical-horizontal figures. The "plus" and square are correctly drawn. The X is made into a "plus", the triangle into a square with a "beak". Complex figures are drawn as squares with elements included.

PERIOD II: INTUITIVE PERIOD (3-4 to 7-8 years)
From square to flag: Mastery of part-whole relations and symmetrical slopes.

Stage IIA: Lower Intuitive (3-4 to 4-5 years). Juxtaposition of unfamiliar elements, globalism.

At stage IIA the subject succeeds at simple figures such as the square, but fails when two new aspects are given: sloped lines and part-whole relations. i) In Figure 5, sloped lines, when not joining two points, are failed (X). Likewise, the top vertices of the triangle and rhombus are drawn as "beaks" or thin lines "added" to figures (nos. 7 and 9). There is *centration* on known schemes, such as the square or circle, and the new aspects are simply *juxtaposed* to those already mastered. ii) Part-whole relations, such as found in the flag (no. 12), are treated through *fragmentation* (Fig. 5) or *globalism* (Fig. 4). (Stage IIA, which represents the γ-*phase* of achievement of Period I, would thus be the beginning or α-*phase* of a new period of equilibration, with the subject meeting the environment new aspects which unbalance his scheme. These are sloped lines and part-whole relations.)

Stage IIB: Middle Intuitive (4-5 to 5-6 years). Mastery of the triangle and snowshoe.

These figures introduce new categories in the subject's behavior: slopes, as opposed to vertical-horizontal lines; or a succession of convex and concave curves in the snowshoe, as opposed to a regular curve. At stage IIB the subject masters these figures. (Differentiation of these two types of behavior would correspond to a β-*phase*, or *categorial differentiation* of behavior.)

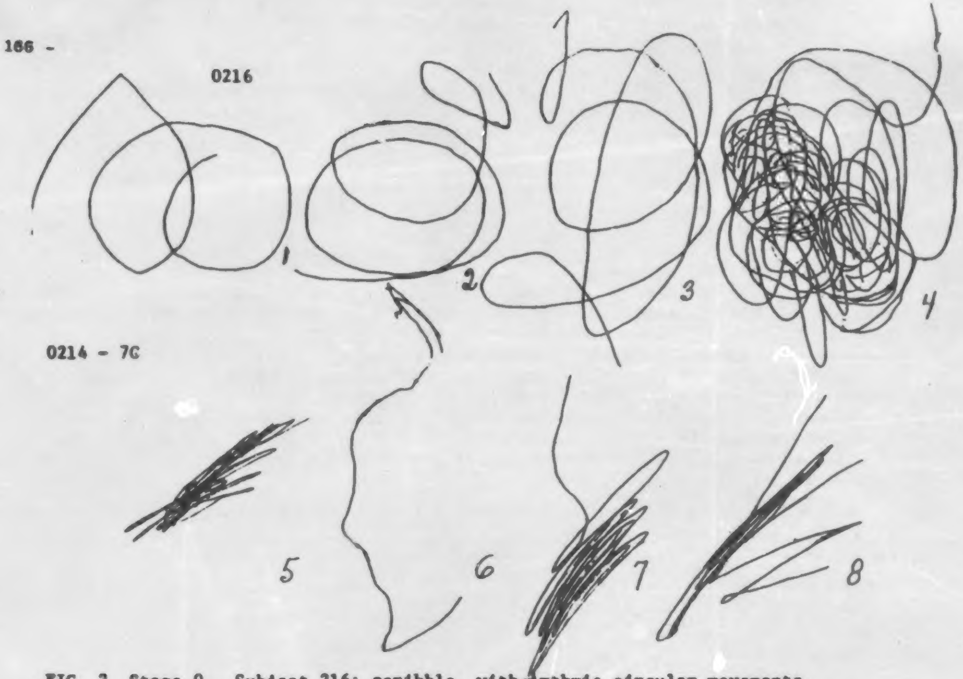

FIG. 2 Stage 0. Subject 216: scribble, with rhythmic circular movements.
Subject 214: scribble with to and fro movements. (I α-*phase*).

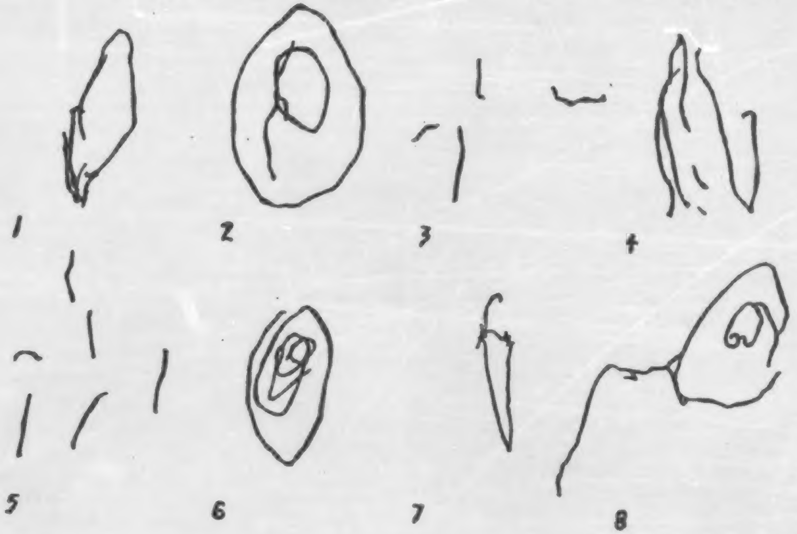

FIG. 3 Stage IA. Subject 304, 3;0 : varied scribble, with closure and collinearity (I β-*phase*).

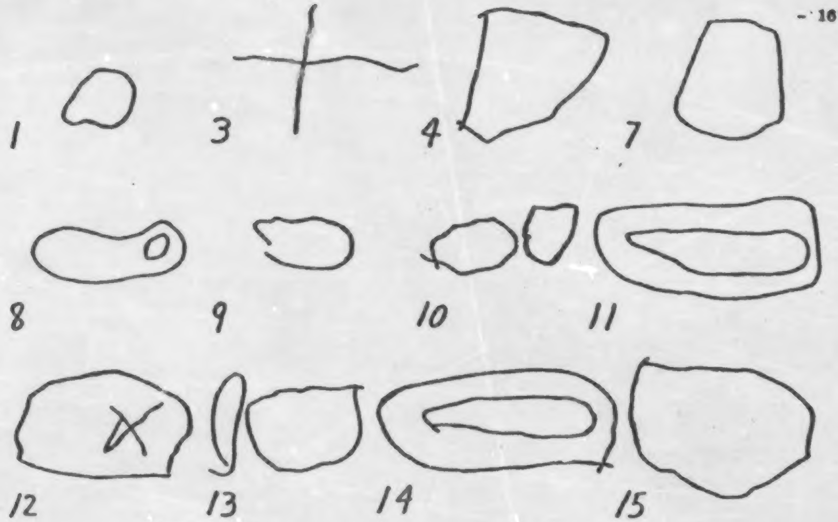
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FIG. 4. Stage IB, FRANCOIS 4;10 : Differentiation of separation and direction (I β-*phase*).

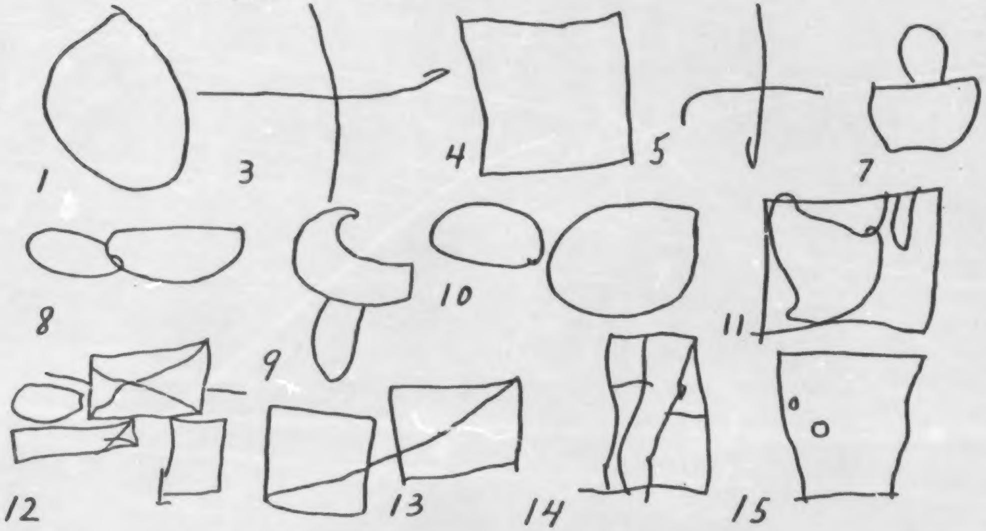
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FIG. 5. Stage IIA. LUCITO 4;0 : Combination of distance and direction leading to success at square (I γ-*phase* → II α-*phase*).

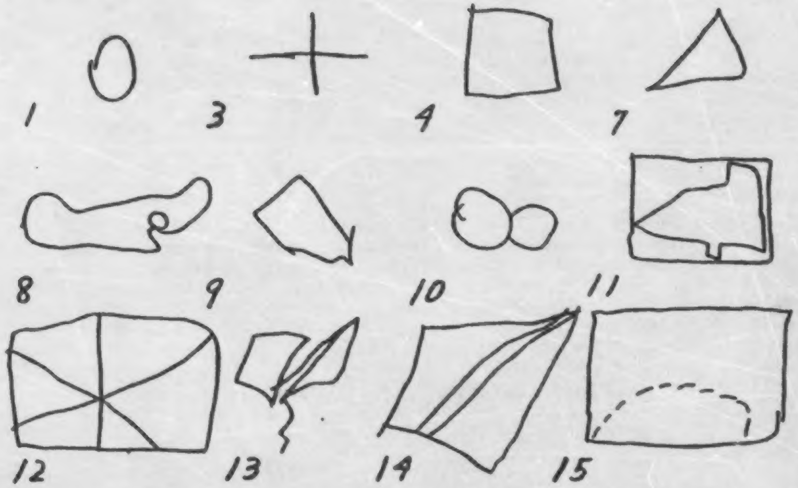
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FIG. 6. Stage IIB. ROBERTO 5;10 : Mastery of inclined segment, globalism (II β-*phase*).

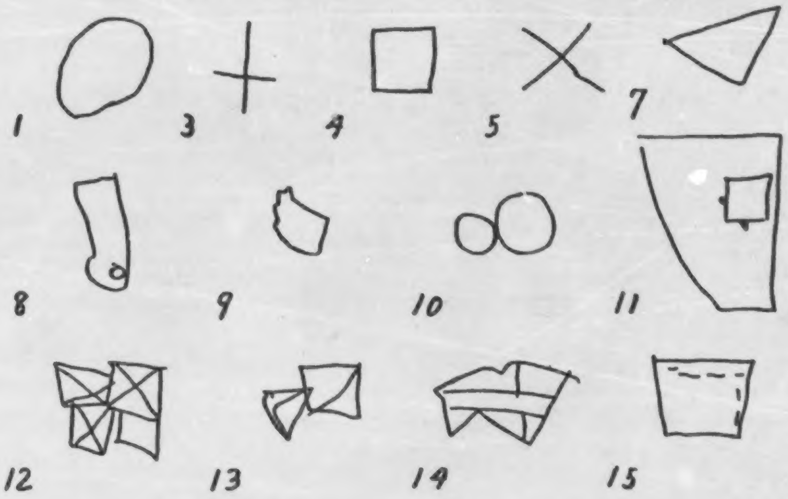
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FIG. 7. Stage IIB. JULIE 5;4 : Mastery of inclined segment, fragmentation (II β-*phase*)

Stage IIC: Higher Intuitive (5-6 to 6-7 years). Mastery of the rhombus.

Mastery of the rhombus involves a *reversal* in the order of sloped lines when passing from the top part to the bottom part of the figure. When starting from a lateral vertex, the top part goes up-down, the bottom part down-up. (This *reversal* would seem characteristic of a *relational* or *ρ-phase* of equilibration. Subjects earlier, instead of reversing, tend to continue in a zig-zag line.)

Stage IIIA: Lower Concrete Operations (6-7 to 8-9 years). Mastery of the inscrived rhombus and flag.

Here *hierarchical figures*, made up both of parts of a whole and symmetrical orientation, are mastered. Parts and the whole are understood as *sharing* their frontiers. A line segment can belong *both* to the part and the whole. Symmetry involves reversal of order. (This stage would correspond to a *γ-phase* of *hierarchical integration*, both of *inclusion* and *direction*. One could consider this stage as that of a *"good gestalt"* in drawing.) Figure 8 illustrates this integration.

PERIOD III: OPERATIONAL PERIOD (7-8 to 16-17 years)

From flag to sigmoid: Introducting curvilinear functions on Cartesian axes.

Stage IIIA: Lower Concrete Operations (6-7 to 8-9 years). Redress of distorting aspects in the figure.

A new aspect which unbalances a "good gestalt" scheme is *distortion*. This is typical of geometrical figures Nos. 13, 14, 15 and 16. Sylvie 7;11 (Fig. 8), who is at stage IIIA, interprets distortion in function of her own schemes. Shifted squares are superposed. The parallelogram becomes a rectangle. (This is characteristic of an *α-phase* of *centration*, here on the "good gestalt". The subject simply omits new cues which would unbalance her schemes.)

Stage IIIB: Higher Concrete Operations (9-10 to 10-11 years). Mastery of perspective in the box.

Distortion of a three-dimensional figure seen in space is here integrated, at a first level, by the "good gestalt" scheme as a *shift* in direction of vertical-horizontal lines. Figure 9 is a good example. Luc 11;7 first interprets a box drawn in perspective as a vertical-horizontal part-whole figure (no. 15a). He gradually coordinates slopes with vertical-horizontal lines and reproduces a recognizable box seen in perspective. The differentiation and coordination between two types of direction: vertical-horizontal and sloping, is characteristic of a *β-phase* of equilibration, or *dualism*. Note that there is no integration yet of directions in a common vanishing point.)

Stage IVA: Lower Formal Operations (11-12 to 14-15 years). Mastery of the staircase.

It is only after 11-12 years that quantification is integrated in perspective, with mastery of the concept of three-dimensional volume. The staircase involves a succession of directional changes. (This corresponds to coordination of *relations* between directions, making it a *ρ-phase* of equilibration.)

Stage IVB: Higher Formal Operations (from 15-16 years onwards). Mastery of mechanical curves.

At various ages, dependent on familiarity with movement of bodies in space, but

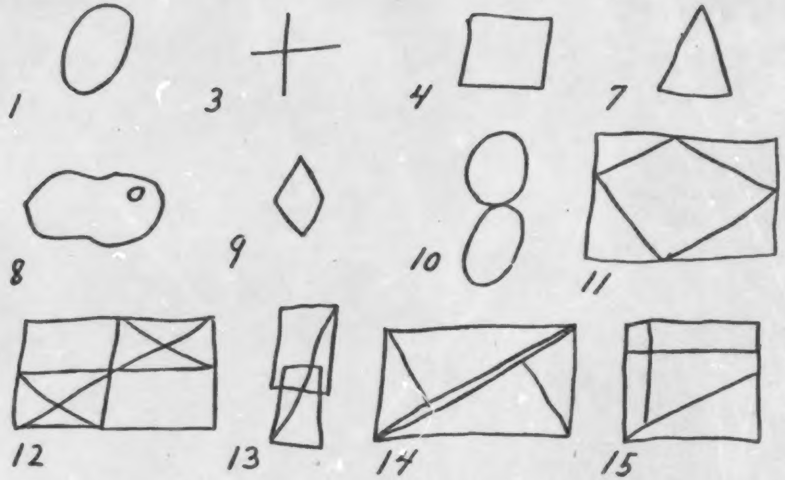

FIG. 8. Stage IIIA. SYLVIE 7;11 : Part-whole integration and symmetry (II *γ-phase* + III *α-phase*).

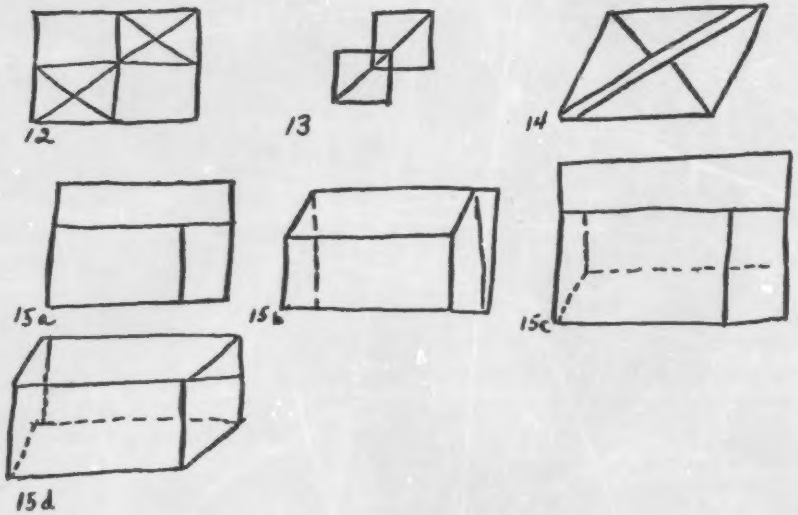

FIG. 9. Stage IIIB. LUC 11;7 : Gradual mastery of perspective through coordination of part-whole relations and direction (III *β-phase*).

not before 15-16 years of age, subjects are able to reproduce mechanical curves after careful experimentation. This involves *differentiation* between various *vertical-horizontal positions* of the square, triangle or circle, as it rolls on its sides, and the corresponding *circular movement* of a vertex or point which rotates with the figure. Correspondence must be established between the Cartesian coordinates of the vertex, seen as successive states in a vertical-horizontal frame and the continuous curve coordinating these states during rotation. (This stage is, of course, an end or γ-*phase* of *hierarchical integration*.)

The operational period began with mastery of vertical-horizontal "good gestalten". Gradually distortion is introduced, and coordinated inside a Cartesian frame. This coordination is made in the form of a "vanishing point" in Experiment 1, or a curvilinear function around Cartesian coordinates in Experiment 2. At the end of Higher Formal Operations, the two original movements found in scribble: circular movements and to and fro motion, are differentiated and coordinated in a structured whole, which we call geometry. This highly condensed summary gives only the outline of the detailed analysis which is being undertaken. Some aspects are still tentative (stage IVA).

CONCLUSION

At the end of this study certain points could be made:

1. Structures should be worked out in a hierarchical manner. The concept of *hierarchy* should be introduced in mathematical education.
2. An accent on *process* should be made, with insistence on *differentiation* and *integration*. Development proceeds through differentiation of schemes to adapt to new reality and integration of the new and old in a structured whole. Thus *order* in which concepts are introduced should be studied.
3. Axioms should stem from *intuition*, which is the first apprehension of reality. Intuition, on which Fishbein (1978) has insisted, should be the ground work of later *axiomatic construction*.
4. At all stages, *rectilinear* and *curvilinear* forms are constructed, with relations among them. Education should take this into account. Triangles should be thought of early as inscribed in a circle, with methods for finding the center. The Pythagoras theorem should not be "parachuted", but explained in terms of relations between linear and plane dimensions. Thus relations between rectilinear and curvilinear figures,on the one hand, and linear and bidimensional concepts, on the other, should be established early.
5. At all levels, intelligence works either in a *holistic* or *atomistic* manner. Freudenthal (1978) has insisted on this holistic view. It should be integrated in education, where atomism has prevailed for too long.

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Mathematics Education Group
Brunel University, Department of Education

Investigations Into The Mathematical Difficulties Experienced By Students And the Reason For These Difficulties

Researchers: Dr. Ruth Rees, Mr. Jim Curnyn, Mr. George Barr.

0. Background

The studies being undertaken are directed to investigating the mathematical difficulties experienced by many different kinds of student, the reasons for these difficulties and suggestions for teaching strategies.

The research method involves a combination of the psychometric and case-study approaches.

1. Samples

Primary	Fourth Year Junior
Secondary	CSE, 'O' and 'A' Level
F.E.	Craft and Technician, ONC.
College of Education	Second year students, on a 3 year course including 'mature'students. Inservice F.E. business studies lecturers.
University	First year engineering undergraduates.

2. Some Items and Concepts Which Are Common To All The Studies

2.1 Pupils At The End Of Primary Education (N=300)

These items, although rated as relatively 'easy' on an 'easy-difficult' rating scale had low facility values for this sample.

Teachers also tended to underestimate the difficulty experienced by pupils with these items.

	Facility
1) What number must go in the box to make this true? $\frac{\square}{8} = \frac{3}{12}$	38%
2) $0.3 \times 0.3 =$	9%
3) Write $\frac{3}{8}$ as a decimal	10%
4) $513 \div 27 =$	32%
5) Write $\frac{37}{100}$ as a decimal	42%
6) How many squares with sides of length 3 cm could I fit into a square with sides of length 6 cm?	39%

2.2 Fifth Form Pupils Taking CSE and 'O' Level

Some Facilities for selected Items

	Facility for:	
	'O' Level Group ($N = 167$)	CSE and 'O' Level Mix ($N_{CSE} = 152$, $N_{'O'} = 62$)
1. $513 \div 27$	92%	50%
2. 0.3×0.3	62%	44%
3. $0.7 \div 0.7$	73%	51%
4. $\frac{1}{x} = \frac{3}{4}$, x is:	60%	28%

2.3 Sixth Form Pupils Taking 'O' Level (N=50)

A reflection of their ideas about number. Consider the following:

	Correct	Popular Response
1. $3 + 6 \times 7 - 5$	40 (2%)	58 (82%)
2. 0.3×0.3	0.09 (64%)	0.9 (34%)
3. $0.7 \div 0.7$	1.0 (41%)	0.1 (36%)
4. $\frac{1}{x} = \frac{3}{4}$, x is:	$\frac{4}{3}$ (43%)	$\frac{3}{4}$ (27%)
5. If $y + 6 = 2y - 7$ then y is equal to:	13 (57%)	-13 (16%)

2.4 TEC Level I Students (N=127)

	Correct	Popular Response
1. $3 + 6 \times 7 - 5$	40 (19%)	58 (71%)
2. 0.3×0.3	0.09 (59%)	0.9 (39%)
3. $0.7 \div 0.7$	1.0 (62%)	0.1 (19%)
4. $\frac{1}{x} = \frac{3}{4}$, x is:	$\frac{4}{3}$ (46%)	$\frac{3}{4}$ (16%)
5. If $y + 6 = 2y - 7$ then y is equal to:	13 (55%)	-13 (12%)

3. Interview Transcripts (End of Primary)

Interviewer: "What is six divided by six?"
Pupil: "One."
Interviewer: "Three divided by three?"
Pupil: "One."
Interviewer: "Sixty two divided by sixty two?"
Pupil: "One!"
Interviewer: "Two thousand and twenty two divided by two thousand and twenty two?"
Pupil: "One!!"
Interviewer: "One half divided by one half?"
Pupil: "A half!......Point five!"
Interviewer: "One quarter divided by one quarter?"
Pupil: "A quarter!"

$0.4 \times 0.4 =$

Pupil: "Well....you keep the point in the same place....goes.... four fours are sixteen....is...one point six."
Interviewer: "Why did you decide to put the one on that side of the point?"
Pupil: "Well....in the same line...points always should be really."

4. Language Laboratory Recordings (School Pupils Aged 13)

If $\frac{1}{x} = \frac{3}{4}$ then x is:

a. $\frac{3}{4}$ b. 3 c. 4 d. $\frac{4}{3}$ e.

(Space e was left for the student's own solution if required.)

1. $\frac{1}{x} = \frac{3}{4}$ then x is .....$\frac{3}{4}$, good. No. Must work this out $\frac{1}{x}$. No, can't work it out... it's something....uh...uh, do this one.... I think....I think.... Oh, I don't know... leave this one....Go back to this in a minute. No.... in a minute.
2. ......, then $\frac{1}{x}$ is....um....um....(long pause)....uh (sighs). If $\frac{3}{4} = 1$..... then x is.....$\frac{1}{12}$ because....because...3...3 x 1 are 3 and 4 x 1 are 4 so if....oh, x = 1.

5. Language Laboratory Recordings (Brunel Engineering Undergraduates)

If $\frac{1}{x} = \frac{3}{4}$ then x is:

a. $\frac{3}{4}$ b. 3 c. 4 d. $\frac{4}{3}$ e.

(Space e was left for the student's own solution if required.)

1. ....Just a very simple....multiply through-out by x and 3x equals 4. Therefore x equals $\frac{4}{3}$ ....(d).
2. ...Dear, dear, my mind is usual blank with this...pause,..oh..of course it's.....um.....yes sorry.... I remember a moment to write that out, knew relationship $\frac{4}{3}$.

6. Teachers' Comments

Teacher 1 Numbers less than one:

a) My experience is that once they have learnt anything in their primary school only daily use of something different will eradicate it from their minds. "It gets bigger when you multiply" is like bindweed. You get rid of it and it still keeps cropping up all over the place.

b) $\boxed{\frac{1}{x}}$: I have however realised that we do not actually teach them what to do about $\frac{1}{x} =$ ....except that we deal with it in passing when it crops up in calculations on similar figures and in handling 'distance = speed x time'.

Teacher 2 Numbers less than unity:

I have to admit that, I had not really considered this as a separate problem from 'mixed decimals', that is, it had not dawned on me that 1.6 x 1.6 has less attendant difficulties than 0.6 x 0.6. Occasionally I have been called to a desk and been asked: "Where have I gone wrong?", when a correct answer has been obtained. Further questioning has produced: "I have multiplied, but I have got a smaller answer than the number I started with." This has given rise to a short class-discussion on what happens when we multiply by numbers less than one, but I have never set out to teach this explicitly in the context of decimal fractions.

BRUNEL UNIVERSITY, DEPARTMENT OF EDUCATION

MATHEMATICS STUDY "THE DIAGNOSIS OF SOME BASIC MATHEMATICAL DIFFICULTIES EXPERIENCED BY STUDENTS".

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xvii) "The Structure of Mathematical Abilities: Some Examples Illustrating Dimensions of Difficulty". Paper given at the Second International Conference for the Psychology of Mathematics Education, Osnabrueck, September 1978.

* (Available from Brunel University Bookshop or Department of Education. Cost £1.50.)
** (Available from Brunel Department of Education. Cost 50p.)
*** (Available from The Institute of Mathematics and its Applications.)

THE IMPORTANCE OF 'ACTION' AND DOING IN THE LEARNING PROCESS.

Dr. D.M. ROBINSON.

"Action" Piaget states "is the genesis of thought". Both this statement and his work illustrating the stages in development of thinking in the child have contributed to and apparently support the new emphasis on practical work as the answer to many of the problems in the teaching and learning of mathematics. Even as prominent a mathematical educator as F.P.Dienes advocates the use of cleverly designed and highly structured practical apparatus. Yet it is notable that Piaget has said very little about the implication of his work for the teacher. Action of course to Piaget does not mean doing or mere physical action but incorporates also the parallel interiorised action in the mind.

This study, in a small way, examines the importance of practical work and where better to do this than with children in the concrete stage. The topic of fractions was chosen as one which lends itself to a practical approach and it seemed not only interesting but also meaningful to compare a very practical approach with a more traditional book-based approach.

EXPERIMENTAL DETAILS.

Two mixed ability third year classes were chosen from a volunteer local junior school. At first it was proposed to match if possible about sixteen children from each class, to teach one group practically, the other traditionally, then to compare their understanding of and their ability to use their knowledge of fractions. But this would have been, to some extent, a false situation, so it was decided to teach complete classes. At the end the matched groups would be compared.

Different people mean different things when they talk about traditional and practical methods. It therefore seems important to define or describe the two types of appraoch to be used in this experiment. The approach used in the Beta books was chosen as a fairly typical traditional teaching method. Deciding on a practical approach however was rather more difficult. In the first place there is a view misguidedly held by some that children will learn just by being allowed to play with and explore apparatus. At least two pieces of research in the United States have shown this view to be fallacious.

In the second place the use of apparatus in 'structured play' situations is not enough. It is not just using apparatus which is important, but the recording and the meaning of the recording. For the purposes of this experiment then it was proposed to ensure that the children spent one third to one half of their time on practical work, the rest being equally split between developing recording skills and doing practical problems.

As no suitable commercial materials were available the workcards which the practically taught class used were produced by me. In order to simulate a realistic teaching situation I based these as far as possible on what a teacher could or would actually do. They involved the use of string, card, paper plates, plasticine, weights, water and sand. Typical workcards for each class are attached.

THE TESTING.

(a) Before the experiment.

There are a wide range and variety of tests available for determining the mathematical attainments of children. The Leicester Number Test was chosen because it is designed to indicate areas of weakness in 7 and 8 year old children and to cover the simple mathematical concepts of counting, number equivalence, number sequence and the operations of addition, subtraction, multiplication and division. Unfortunately the numerical scores were not sufficient to grade the children as it was quite possible for two children, one with a good understanding of the basic concepts, and the other with a poor understanding but able to learn and remember rules to obtain exactly the same scores. In this respect then the test was unsuitable. However close examination of the results made it possible to grade the results into the categories:

A - very good
B - good
C - average
D - poor.

These were based primarily on good cardinal and ordinal number concepts together with good single digit operation concepts and skills. (Skills and abilities necessary to handle the fraction work to be done) - see Tables 1 and 2 for further detail. These categories enabled a better matching to be made than the raw scores. As a check on the correctness and reasonableness of the matching the N.F.E.R. verbal reasoning and the Burt reading tests were employed. See Table 3.

Finally a simple test was given to the children to check the assertions of the teachers that both classes had little knowledge of the fractions. (On the whole the classes seemed comparable both sets of children knew the names of and could record simple fractions such as $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{3}{4}$. But they could not perform recorded mathematical operations such as $\frac{1}{4} + \frac{1}{4} =$; $3 \times \frac{1}{4} =$; $\frac{1}{2} \times 4 =$ or $3 \div \frac{1}{2} =$

(b) After the experiment.

The experiment lasted four weeks and involved a one hour period on three successive days in each of these weeks. In all twelve hours work. During this period the children were taught four topics.

(i) Wholes and equal parts.
(ii) Equivalent fractions.
(iii) Mixed numbers.
(iv) Addition and subtraction.

Final testing covered each of these topics and each test was so constructed as to examine understanding, recording skills and problem solving. These tests were designed to be straightforward and examine the children's knowledge of what they had been taught. (As such they dealt in the main with simple fractions such as halves, quarters, eighths, thirds and sixths)

In order to highlight differences between the classes another set of tests were devised. These were designed to assess the children's ability to transfer and use their knowledge in new or different situations. The following areas were investigated.

(i) Their knowledge and understanding of what a fraction is. this involved a number of simple examples of situations different to anything they had met before.

(ii) Their ability to deal with multiple wholes. Till now they had only dealt with fractions of 1 whole unit.

(iii) Their ability to handle fractions in the new situation of the number line.

(iv) Their grasp of fractional size and fractions of a fraction.

THE RESULTS OF THE TESTS.

TABLE 3. This shows the initial test results for the children who were chosen as matched pairs.

TABLE 4. This gives the results for the tests on knowledge of the taught material together with the results of the tests on transfer.

TABLE 5. This table gives the breakdown of the tests on knowledge of the taught material into the areas of concepts, recording skills and problem solving.

TABLE 6. This table gives the results of an attitude test and of a retention test which was given 6 weeks later.

1. All ability groups, A to D, showed a superiority in all tests for the traditionally taught children.

2. Student t tests were used to compare the results of the two classes. At the 5% level there was no significant difference for the tests on knowledge of the taught material. However the tests of transfer gave a significant result of 2.24 against a table value of 2.

3. The results on concepts, skills and problem solving showed no great increase for the practically taught children. On the contrary the lower ability groups C and D appear to have suffered - a surprising result.

4. The attitudes for both classes towards the work and the teaching were comparable and quite good. NO significant difference.

5. After six weeks the average loss for the practically taught children was 14% as opposed to 7% for the traditionally taught. (This result however was not significant)

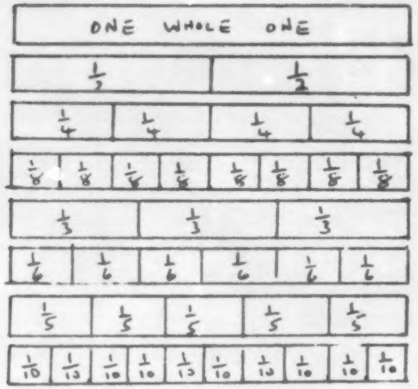
CONCLUSIONS.

The purpose of this experiment was to investigate the importance of doing (physical action) on the consequent parallel mental action which necessarily occurs in the learning process. The results of this small experiment dealing with nine and ten year olds most of whom are in the concrete stage have raised some serious questions as to the importance and amount of physical action that is really required. In particular they seem to suggest that meaningful mental action and the corresponding understanding can often come from good easily perceived visual illustrations. Further research in this area is obviously needed.

Typical Traditional Workcard

Fractions

Look carefully at the diagram below and then answer the questions.



C Write the following fractions, putting in the missing number

1 $\frac{1}{4}=\frac{\square}{8}$ 2 $\frac{1}{5}=\frac{2}{\square}$ 3 $\frac{3}{8}=\frac{\square}{16}$ 4 $\frac{1}{2}=\frac{\square}{8}$ 5 $\frac{2}{5}=\frac{4}{\square}$ 6 $\frac{1}{2}=\frac{\square}{16}$ 7 $\frac{1}{4}=\frac{\square}{16}$ 8 $\frac{5}{8}=\frac{10}{\square}$

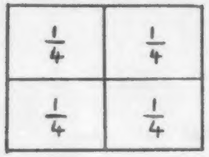
Look at each answer. You see that if the top and the bottom of the fraction are multiplied by the same number, the value of the fraction is unchanged.

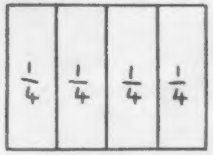
9 $\frac{2}{4}=\frac{1}{\square}$ 10 $\frac{8}{16}=\frac{\square}{2}$ 11 $\frac{6}{8}=\frac{\square}{4}$ 12 $\frac{8}{10}=\frac{4}{\square}$ 13 $\frac{2}{8}=\frac{1}{\square}$ 14 $\frac{4}{10}=\frac{2}{\square}$ 15 $\frac{2}{4}=\frac{\square}{2}$ 16 $\frac{8}{16}=\frac{\square}{2}$

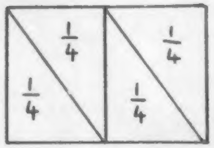
Look at each answer. You see that if the top and the bottom of the fraction are divided by the same number, the value of the fraction is unchanged. This is called cancelling.

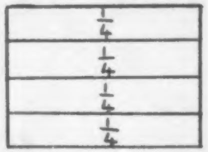
WORKCARD F.1. Typical Practical Workcard

1. Use 4 equal paper rectangles.
   Fold them to make these.









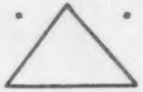
Cut out four different looking quarters and stick them in your book. Write on each what fraction it is.

2. Take 3 equal rectangles. Fold them in half in different ways. Cut out 3 different looking halves. Stick them in your book. Write their names on them.

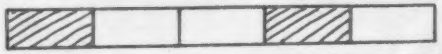
Test on Knowledge

Wholes and equal parts

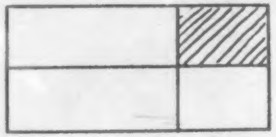
1. Draw a half of this shape



2. What fraction is shaded?



3. Is the shaded part a $\frac{1}{4}$ of the shape?



4. Draw a circle and shade "$\frac{1}{2}$"

5. Draw a square and shade "$\frac{3}{4}$"

6. Draw a circle and shade "$\frac{5}{8}$"

7. A bar of chocolate is shared equally between John, Paul and Peter. What fraction does John get?

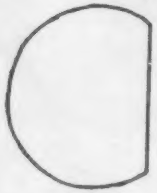
8. A whole cake is shared equally between six people. How much does each get?

9. Mary and Ann share an ice-cream. If Mary eats twice as much as Ann what fraction does Mary eat?

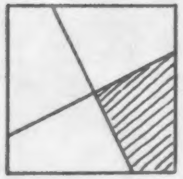
Test 1

(Transfer)

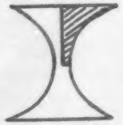
1. Shade $\frac{1}{2}$ of this shape.



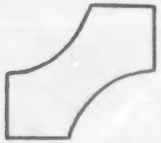
2. What fraction is shaded?



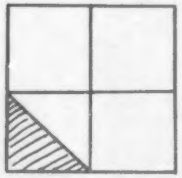
3. What fraction is shaded?



4. Shade $\frac{3}{4}$ of this shape.



5. What fraction is shaded?



6. What fraction are men?

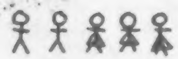


Table 1. (Practical class)

	Classif-ication	Score	Areas of Weakness: Number concept	Operations + - x ÷	Borrowing (tens & units)	Vertical x & ÷	Pattern
Sharon	B	66				X	
Nicola	C	62				X	X
Lee	B	64				X	X
Timothy	C	42		X	X	X	X
Paul	B	63				X	
Anita	D	58		X	X	X	X
Adrian	B	60				X	
Simon	A	73					
Lisa	B	66				X	
Teresa	C	63			X	X	X
Elaine		26	X	X	X	X	X
Ian	A?	68					
Mark	B	67				?	
Peter	D-	49	X	X	X	X	X
Jonathan	A-	67					X
Hayley	D	56		X	X	X	X
Garry	C	52		X	X	X	X
Sharon 2	C	63			X	X	X
Sari	-	-					
Tracey	C	65		?	X	X	X
Linda	D	43		X	X	X	X
Darryl	C	62		(X?)	X	X	X
Saonie	C-	57			X	X	
Jason	A	72					
Jayne	C+	64				X	X
Mark 2	A-	69					X
Darren	C	52			^	X	X
Deborah	C	65				X	X
Sarah	A	66					

X means weak in these areas.

Table 2. (Traditional class)

	Classif-ication	Score	Areas of Weakness: Number concept	Operations + - x ÷	Borrowing (tens & units)	Vertical x & ÷	Pattern
Katie	A	70			X	X	
Alan	B					?	X
Paula	D?	49		?	X	X	X
Nicola	A	69					
Sharon	B	68				X	
Andrew		60			•	•	•
Ray	A	65					X
Lyn	C+	62		?÷	X	X	
Gareth	C	62			X		X
Robert	D	54		X	X	X	X
Louise	B-	62				X	X
Shaun	C	59			X	X	X
Vicky	C	64		?÷	X	?÷	X
Jackie	C	65		?÷		X	X
Jason	C	66			X		X
Rachel	C-/D	60			•	•	•
Nathalie	B	68				X	
Mark	A	73					
Darren	B	68			X	X	
Garry		45			•	•	•
Pat	A	67					
Amanda	B+	68					X
Bernice	B	66			X	X	X
Mark 2	D	49		X	X	X	X
Joanna	A	66					
Simon	C	59				X	X
Tracey	C	61				X	X
Martin	C	39			•	•	•
Jessie	B	67				X	
Mark 3	C+	58			X	X	
Neil		57					
Colin	B	63			X		

Table 3.

Leicester Number Test Grade	Leicester Number Test Score	N.F.E.R. Score	Burt Reading Age
		Traditional	
A	70	122	10:11
B	68	117	11: 2
A	69	107	9:11
A	65	110	10: 7
A	67	104	9: 1
B	67	98	8:10
C	66	103	9:10
B	63	112	11: 5
B	68	92	8: 4
C	62	90	9: 0
C+	62	99	9: 9
C	59	98	10:11
C	61	100	8: 8
D	49	97	9: 0
D	45	77	7: 2
C	59	94	10: 9
		Practical	
A	73	121	12
A	72	120	10:11
A-	69	107	10: 2
A-	67	99	9: 7
A-	68	101	9: 6
B	64	97	9: 5
C+	64	103	9:10
B	67	105	8:11
B	63	96	10: 2
C	63	90	9: 5
C	62	93	9: 5
C-	57	91	8: 1
C	63	103	8: 7
D	49	96	10: 3
D	43	80	7:11
D	58	85	8: 5

Table 4.

Results of the tests on knowledge

These were the tests on Wholes, Equivalence, Mixed Numbers and Addition.

Groups	Practical Marks				Practical Total	Traditional Marks				Traditional Total
A	9	8	9	8	34	9	9	9	8	35
	9	9	9	8	35	8	9	9	7	33
	9	8	9	7	33	9	8	9	8	34
A-	9	7	9	8	33	9	6	9	8	34
	8	8	9	7	32	8	7	9	7	31
B	8	8	9	8	33	7	7	9	8	31
	9	8	8	6	31	7	9	7	7	30
	9	8	9	8	34	9	8	9	8	34
	4	5	8	4	21	8	9	8	7	32
C	3	3	2	2	10	6	6	8	4	24
	8	6	9	5	27	8	5	8	8	29
	6	4	5	3	18	6	4	6	6	22
	7	7	8	9	31	7	8	9	7	31
D	3	3	7	1	14	7	5	7	3	22
	3	2	5	0	10	3	1	3	2	9
	4	5	5	4	18	5	4	5	1	15

Results of the tests on transfer

These were tests T.1., T.2., T.3. and T.4.

Groups	Practical Marks				Practical Total	Traditional Marks				Traditional Total
A	6	6	7	6	25	6	6	6	6	24
	4	6	7	6	23	6	6	7	5	24
	4	6	6	6	22	6	6	6	6	24
A-	4	5	6	3	18	5	6	7	6	24
	5	5	7	6	23	6	6	5	5	22
B	5	5	4	5	19	6	4	7	5	22
	5	5	5	5	20	5	5	7	6	23
	5	6	5	6	22	6	5	7	6	24
	4	3	1	6	14	6	5	7	5	23
C	5	1	0	0	6	4	6	1	4	15
	5	6	6	6	23	6	5	5	5	21
	4	3	1	0	8	6	6	5	5	22
	4	6	4	6	20	3	5	3	3	14
D	2	3	1	2	8	2	4	2	3	11
	4	1	0	1	6	6	3	2	3	14
	1	3	[illegible]	[illegible]	[illegible]	[illegible]	6	[illegible]	[illegible]	[illegible]

Table 5.

Results on Understanding; Recording; Problem Solving

The first set of tests that is the tests to assess the children's knowledge of the material they had been taught were designed to look at three areas. Each test consisted of nine questions in which the first three dealt with understanding concepts, the next three with skills at recording, and the last three examined the children's ability to relate their knowledge and skills to a real or problem situation. This design and balance was quite deliberate so that it would be possible to compare both the practically taught and the traditionally taught groups in these areas. The following were the results obtained:-

Grades	Practical			Traditional		
	Concepts	Skills	Problems	Concepts	Skills	Problems
A	12	12	10	12	12	11
	12	12	11	11	12	10
	11	11	11	12	12	10
A^-	12	11	10	12	12	10
	12	11	9	11	11	9
B	12	12	9	12	11	8
	11	11	9	8	12	10
	12	12	10	12	11	11
	8	9	4	12	12	8
C	3	6	3	8	7	9
	10	9	9	10	11	8
	3	6	4	8	10	4
	12	10	9	11	1[illegible]	10
D	6	6	2	9	7	6
	3	5	2	5	4	0
	5	9	4	5	6	4

If, for each group, the totals of these results are expressed as percentages the following values are obtained:-

Groups	(Practical)			(Traditional)		
	Concepts	Skills	Problems	Concepts	Skills	Problems
A/A^-	98	95	85	97	98	83
B	90	92	67	92	96	78
C	69	65	52	78	83	65
D	39	56	22	52	47	27

Table 6.

Attitude Testing

The children were asked to express their opinion of the work we had done on fractions (not the testing). A typical five point scale was used for this.

1. Enjoyed it very much
2. Enjoyed it
3. It wasn't bad
4. Didn't like it
5. Thought it awful

Results

	Practical Class					Traditional Class				
Attitude	1	2	3	4	5	1	2	3	4	5
Numbers	5	13	11	0	2	1	15	8	1	1
Average	2.35					Average 2.45				

Testing Retention

After a period of six weeks the tests of the children's knowledge of the taught material were again given to both classes. The following results were obtained:

Groups	Practical class					Traditional class				
	Marks				Total	Marks				Total
A	9	9	8	8	34	9	8	9	8	34
	7	8	8	8	34	8	7	8	6	29
	7	8	9	7	31	8	7	9	7	31
A^-	8	8	9	7	32	8	9	6	8	31
	7	8	9	8	32	8	8	9	4	29
B	7	6	8	6	27	8	7	8	6	29
	9	9	8	6	32	8	9	8	6	31
	9	7	8	5	29	8	8	8	6	30
	6	7	8	7	28	8	9	8	7	32
C	3	2	5	1	11	7	9	7	6	29
	8	7	7	3	25	6	6	6	6	24
	5	3	7	5	20	6	3	2	4	15
	4	4	5	5	18	6	6	7	7	26
D	3	6	4	1	14	5	5	6	3	19
	3	0	0	4	7	2	1	5	3	11
	4	2	1	4	11	7	3	3	4	17

The percentage losses on the total marks for each group are given below:

Groups	Practical class Percentage loss after six weeks	Traditional class Percentage loss after six weeks
A	6	8
A^-	2	8
B	20	5
C	15	12
D	24	-2

THE RELATIONSHIP BETWEEN VISUAL SPATIAL ABILITIES AND MATHEMATICAL PROBLEM SOLVING: ARE THERE SEX-RELATED DIFFERENCES?

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In the United States an important focus of recent research in mathematics education has been sex-related differences in performance, especially on more complex tasks such as problem solving. If such differences are found, or even anticipated, a search is often made for other cognitive abilities which show sex-related differences and which may be related to mathematical problem solving. Research of this type has been substantially funded by federal agencies and has been reported with increasing frequency at professional meetings. Visual spatial abilities have been chosen for further study by some investigators (Fennema & Sherman, 1978; Schonberger, 1976) because the development of sex-related differences in these abilities appeared to parallel or precede their development in mathematics achievement (Fennema, 1974; Maccoby & Jacklin, 1974). MacFarland-Smith (1964) has given evidence of the importance of spatial ability in tests which are genuinely mathematical as distinct from those which are merely mechanical or computational. The use of charts, diagrams and graphs in all branches of mathematics also argues for the logic of this connection.

To impose some order on the chaos of what have been referred to as visual spatial abilities, this author has characterized them as those measured by tests (usually paper-and-pencil) presenting figural stimuli and involving movement or transformation of the figures. The figures may represent two- or three-dimensional objects. Of great use in the ordering of the factor analytic research in this area of cognitive abilities in the United States was the synthesis by Michael, Guilford, Fruchter, and Zimmerman (1957) in which subfactors were identified and described. This synthesis was influential in Guilford's (1967) development of his structure-of-intellect model, and in the development of the Kit of Reference Tests for Cognitive Factors (French, Ekstrom, & Price, 1969) under the auspices of the Educational Testing Service (subsequently referred to as the ETS Kit). Two of the three subfactors identified by Michael et al have been used in the study of mathematical abilities by, among others, the National Longitudinal Study of Mathematical Abilities (NLSMA) and should be discussed in some detail. (It should be noted that not all factor analytic studies of batteries of spatial tests find subfactors, but if the space factor splits, it is usually along these lines.)

The first factor is called Spatial Relations - Orientation (Michael, et al, 1957), cognition of figural systems (Guilford, 1967), or Spatial Orientation (French, et al, 1969). It is described in the ETS Kit as "the ability to perceive spatial patterns or to maintain orientation with respect to objects in space" and is "perhaps best thought of as space with the observer himself as a reference point" (French, et al, pp. 38-39). From a mathematical, rather than psychological, point of view these tests require only rigid transformations of the whole geometric figure (i.e. slides, flips, and turns). A two-dimensional example of an SR-O test is Card Rotations from the ETS Kit which was used in the NLSMA (Romberg & Wilson, 1969). The task is to discriminate between rotations and reflections of a referent figure. (See Figure 1.)

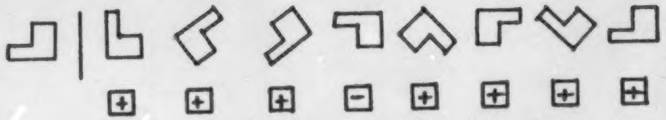

Figure 1. A Set of Card Rotations Items (2-Dimensional SR-O).

A three-dimensional example of an SR-O test is Cubes Comparison (available from ETS, Princeton, N.J.) on which each item presents two drawings of cubes with a letter, numeral or other symbol on each face. Assuming no symbol can appear on two faces of the same cube, the subject is to indicate which pairs of drawings can represent the same cubes and which pairs must represent different cubes. (See Figure 2.)

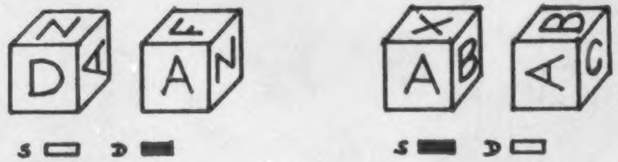

Figure 2. Two Cubes Comparison Items (3-Dimensional SR-O).

These items can be done by rotating one cube and translating it onto the other. Speed is usually quite important in SR-O tests.

The second factor is called Visualization (Vz) in the ETS Kit and is described as "the ability to manipulate or transform the image of spatial patterns into other visual arrangements" (French, et al, p. 47). Tests of this factor require other nonrigid geometric transformations such as cutting, folding, or punching holes as well as rotations, reflections, or translations. A two-dimensional example of a Vz test is Form Board 2, an ETS Kit test modified for use in the NLSMA (Romberg & Wilson, 1969). Each item shows five shaded pieces, some or all of which can be rotated and translated to fit together like puzzle pieces to form the figure presented in outline form. (See Figure 3.)

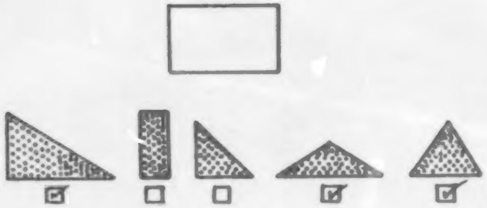

Figure 3. An Item from Form Board 2 (2-Dimensional Vz).

A three-dimensional Vz test is Space Relations from the Differential Aptitude Test battery (DAT) (Bennett, Seashore, & Wesman, 1973) which is like Surface Development, a three-dimensional Vz test from the ETS Kit. The task in DAT Space Relations is to choose which of four drawings represents the figure which would be formed by folding and pasting the paper pattern shown. (See Figure 4.)

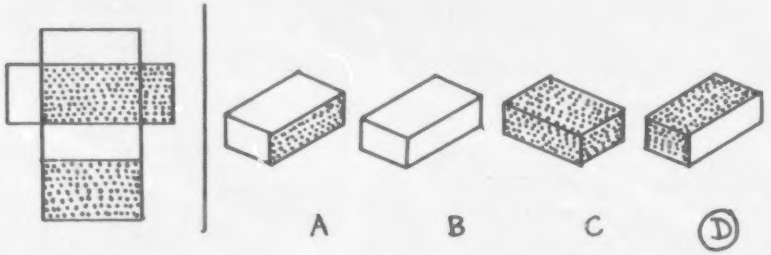

Figure 4. An Item from DAT Space Relations (3-Dimensional Vz).

Examples of spatial problems have been given in the preceding paragraphs, but the term mathematical problem has been applied to a variety of tasks. This author uses Zalewski's (1974) subject-dependent definition of a mathematical problem, which requires that the person trying to solve the problem not possess an immediate answer or procedure for solution. To qualify for consideration here a study must have used test items intended by their designers to measure mathematical abilities other than computation and judged by the author to satisfy Zalewski's definition.

The facts that spatial tests can be classified in terms of the topological transformations they involve and topology can be the subject matter for mathematical problems suggest that the cognitive processes used in solving spatial and mathematical problems might have something in common. Studies of the relationship between these two types of problem solving were reviewed with these questions in mind.

1. What is the evidence of a relationship between mathematical problem solving ability and visual spatial abilities?

2. If more than one measure of spatial skills was given, are the relationships with the test of mathematical problem solving different for different spatial tests?

3. If more than one problem solving measure was used, are they differently related to the test(s) of visual spatial abilities?

4. Are any of the above relationships different for males and females?

Answering the questions about differences in relationships raised a statistical concern which should be discussed. Most of the studies reviewed used correlation coefficients or other statistics based on them such as factor analysis or regression analysis. Even with a decent sized sample (100 boys and 100 girls, for example) the difference between two correlation coefficients has to be about .30 in order for that difference in relationships to be significant at the .05 level. For that reason the review of research was suggestive rather than conclusive. Studies providing information on these questions usually have subjects who are nine or older.

A number of studies provided evidence of a relationship between the ability to solve spatial and mathematical problems which was the first question. The Concept Attainment Abilities (CAA) Project used two different samples of fifth-grade students who took Card Rotations and Space Relations, which is a type of form board test, and several tests judged to consist of mathematical problems (Harris & Harris, 1973). Data from this study and a later study by Meyer (1976) modeled on the CAA Project but using fourth grade students are summarized in Table 1. All the correlation coefficients are significantly different from zero at the .01 level. In a study of middle school students, Fennema and Sherman (1978) found a close relationship (.59 for females, .60 for males) between DAT Space Relations and a test of mathematical problems. The National Longitudinal Study of Mathematical Abilities (NLSMA) used Card Rotations, Form Board 2, and Paper Folding which is a two-dimensional Vz test in studying four levels of mathematical behavior. The highest level, analysis, which required nonroutine application of concepts, seemed the closest to the definition of problem solving. Generally there were no other mathematical tests in the NLSMA battery which were more closely related to the spatial tests than were the analysis tests, and only a few were as closely related (Wilson & Begle, 1972b, summarized in Schonberger, 1976).

Table 1

Correlations Between Spatial and Problem Solving Variables in the CAA Project and Meyer's Related Study

	Necessary Arithmetic Operations	Number Series	Romberg-Wearne Problem Solving 3
CAA Project: 1970 Sample*			
Card Rotations			
Girls[A]	.44	.45	
Boys[B]	.37	.36	
Space Relations			
Girls[A]	.41	.45	
Boys[B]	.42	.45	
CAA Project: 1971 Sample*			
Space Relations			
Girls[C]		.51	
Boys[C]		.37	
Meyer's 1976 Study**			
Space Relations			
Girls[D]		.53	.35
Boys[E]		.43	.26

[A] N=210 [B] N=172 [C] N=204 [D] N=82 [E] N=97

*Data from Harris & Harris, 1973.
**Data from Meyer, 1976.

The Project Talent study (Flanagan, Davis, Dailey, Shaycroft, Orr, Goldberg & Neyman, 1964) presented data for twelfth-grade students who took a two-dimensional SR-O test like Card Rotations and a three-dimensional Vz test like DAT Space Relations as well as problem solving tests of three levels of difficulty. This study sampled all subject areas; there were no other tests which were more closely related to the spatial tests than the mathematics tests although some in science were as close. Werdelin's (1958, 1961) factor analytic studies of Swedish high school students also support the close relationship between solving spatial and mathematical problems. In the 1958 study, which had an all-male sample, the Space factor included two tests of mathematical problems, and the Mathematical Reasoning factor included four tests of mathematical problems and all the Vz tests, both two- and three-dimensional. In the 1961 study the General Reasoning factor included the Cubes test as well as all the tests with mathematical problems. All four geometry tests and Number Series had significant loadings on the Space factor. In summary, visual spatial abilities appear to account for part of the variance in mathematical problem solving but the NLSMA and Project Talent data indicate that that part appears to decrease as the subjects get older or the mathematics measure depends more on specific mathematical information. In most cases spatial tests were as closely related to problem solving measures as they were to any other tests.

The second question dealt with the comparison of different types of spatial tests. Although none of the differences were statistically significant, these trends were observed. In the NLSMA data, Paper Folding was a better predictor of the analysis scales than was Form Board 2 which, in turn, was a better predictor than Card Rotations (Wilson & Begle, 1972a, 1972b, summarized in Schonberger, 1976). These are all two-dimensional tests; the first two are Vz, the last is SR-O. In the Project Talent study the three-dimensional Vz test was a better predictor than the two-dimensional SR-O test. In Heseman's (1976) study, which used four tests like those described earlier, the three-dimensional tests had more weight in the regression equation than the two-dimensional tests. From the available data it was not possible to tell whether the Vz, SR-O classification or the dimensional one was most important.

The third question dealt with the types of mathematical problems used. In the NLSMA, problems were categorized as algebra, geometry, or number systems (this last category included most of the real-life problems). The spatial variables were more closely related to the geometry scales and number systems scales than to the algebra scales. These results were substantiated by Dodson's (1972) discriminant analysis of a subset of the NLSMA data for eleventh-grade students.

Whether or not there are sex-related differences in the relationships between the spatial and mathematical variables is a harder question to answer. In the three studies of elementary school children summarized in Table 1 (Harris & Harris, 1973; Meyer 1976), the coefficients for girls are usually higher than those for boys although the largest difference is statistically significant only at the .08 level. In addition, in these studies the spatial tests behaved differently in the factor analyses for girls than for boys; only for girls the spatial tests loaded on the numerical factor. Perl (1979) used the NLSMA data for Grades 10-12 to compute multiple correlation coefficients between nine achievement measures (none computation) and each of two spatial tests. For Card Rotations the girls' coefficient was significantly higher than that of the boys, but for Paper Folding there was no significant difference. Sex-related differences in the correlations between spatial and mathematical variables were not found in the Project Talent study nor in the studies by Fennema and Sherman and by Heseman. Neither Dodson nor Werdelin had separate analyses for males and females.

Even with strong correlational evidence that spatial and mathematical problem solving have much in common, the cause and effect relationships remain hidden. Does one use spatial skills in solving mathematical problems or mathematical skills in solving spatial items? The NLSMA found that a year's training in geometry did not increase scores on Card Rotations (Romberg &

Wilson, 1969), and it is usually assumed in the literature that spatial ability is somehow more elemental than the ability to solve mathematical problems which has other components. To answer the question of how spatial skills are used, one has to turn to introspective as well as experimental research.

Hadamard's (1945) account of his own thinking and that of Einstein suggests that visual imagery plays the role of recording the relationships or patterns among elements of the problem and facilitating the combination of elements into new patterns. However, introspective research on both creative mathematicians (Hadamard, 1945; Poincaré, 1929) and ordinary people (Menchinskaya, 1946/1969) indicates both visual and nonvisual styles of problem solving which have nothing to do with problem content. In addition to the use of mental images, problem solvers often use diagrams, visual images externalized on paper. The ability to draw a diagram correctly representing the information was found useful in solving problems by Botsmanova (1960/1970). Kilpatrick (1969) did not find drawing a diagram significantly related to success in problem solving, but he did not note whether or not the diagram correctly represented the problem. Although sex-related differences were not discussed in the Soviet literature, there is some evidence from other countries that females perform less well at drawing diagrams than males. Michelmore (1975) found fifth-grade girls less able to draw three-dimensional solids from models than boys in two Jamaican studies, but did not find this difference appearing in this country until ninth grade. Boe (1968) found girls in Grades 8, 10, and 12 to be less able to *draw* plane sections of solids but equally able to *choose* the correct diagram of the section.

Could it be that the link between visual spatial abilities and solving mathematical problems is the ability to represent problems by diagrams? The author's study (Schonberger, 1976) was designed to investigate this question along with the four posed earlier. The entire seventh grade (87 boys and 83 girls) of a junior high school in Bangor, Maine, participated. A matrix of spatial tests was chosen to investigate whether it was the SR-O, Vz subdivision, or the dimensional division that was important. (See Figure 5.)

	SR-O	Vz
Two-dimensional	Card Rotations	Form Board 2
Three-dimensional	Cubes Comparison	DAT Space Relations

Figure 5. A Matrix of Spatial Tests.

In order to provide further evidence on the importance of type of problem, problems from Zalewski's (1974) item bank were categorized in the following way:

Type A: Problems in which the stimulus (presentation of the problem) is partly pictorial or which require spatial or geometric skills or knowledge for solution.
Example: The perimeter of (distance around) a swimming pool in the shape of a rectangle is 148 feet. If the length of the pool is 50 feet, how many square feet of surface does the pool have?

Type B: Problems with a complete verbal stimulus in which spatial skills (such as visualizing the situation or drawing a diagram) may be useful but are not necessary.
Example: If six bushels of wheat will plant four acres, how many bushels of wheat are needed to plant 30 acres?

Type C: Problems which appear to have no spatial content. (In other words, any problems that do not fit Categories A or B should be labeled C.)
Example: Jim left Bangor at 3:20 and took one hour and fifty minutes to drive to Portland. What time did he arrive in Portland?

The problem solving test and subtests had three scoring scales: number of correct solutions, number of problems for which diagrams were drawn or modified, and number of problems for which diagrams were drawn or modified to *correctly* represent the information in the problem. Statistical techniques used to study the relationships were correlation coefficients, factor analysis, regression analysis, and canonical correlation analysis.

The results of these analyses agree with some of the results of the other studies reviewed but disagree with others. As in all the other studies, visual spatial abilities were significantly related to problem solving in mathematics for the whole group. The relative importance of Vz and SR-O tests or two- and three-dimensional tests in predicting mathematical problem solving ability was examined using coefficients from the correlation matrix. (See Table 2.) The three-dimensional tests of both types were somewhat better

Table 2

Correlations Between the Spatial Variables and Problem Solutions Total for the Whole Group in Schonberger's 1976 Study

	SR-O	Vz	t Ratio[a] (SR-O - Vz)
2-Dimensional	.350	.234	1.41
3-Dimensional	.395	.446	−.76
t Ratio (3-D - 2-D)	.57	2.72*	

[a] t Ratio for the significance of the difference between correlation coefficients, correlated data. (Downie & Heath, 1970)

*$p < .01$

predictors of the total problem-solutions scale, but the difference was significant only for the Vz tests. The same patterns were observed in the solutions subscales. The surprising result of the investigation related to different types of problem content was that the spatial tests were usually most closely related to solving the B-type problems. This showed up in the correlation coefficients, in the factor analyses where Problems Solutions B loaded on the Space factor and in the regression analyses where the coefficients of determination of the spatial variables were larger for Problem Solutions B than for A or C.

The fourth question was whether or not the relationships between solving spatial and mathematical problems were different for males and females. As shown in Table 3 the correlation coefficients between the spatial tests and the problem solution scales indicate that with two exceptions the two types of variables were more closely related for females than for males. Testing for significance of the difference between correlation coefficients turned up none whose *p*-level was less than .05 but a number of the *Z* values were greater than one. In the regression analyses, the percent of the variance in the females' scores accounted for by the spatial variables was from 7 to 18 percentage points higher than in the males' scores.

Table 3

Correlations Between Spatial and Problem Solving Variables in Schonberger's 1976 Study

	Card Rotations		Form Board 2		Cubes Comparison		DAT Space Relations	
	Girls[a]	Boys[b]	Girls	Boys	Girls	Boys	Girls	Boys
Problem Solutions A	.33	.19	.24	.17	.40	.13	.38	.28
Problem Solutions B	.40	.35	.34	.18	.49	.33	.47	.44
Problem Solutions C	.23	.25	.24	.07	.28	.32	.43	.22
Problem Solutions T (Total)	.39	.32	.33	.16	.47	.34	.52	.37

[a] n = 83 [b] n = 87

This was summed up by the results of the canonical correlation which used the three solution subscales as the dependent variables and the four spatial tests as the independent variables. (See Table 4.) The canonical correlation accounted for a larger percentage of the females' variance than of the males' variance. For females scores for both A- and B-type problems were related to that part of the variance predicted by the spatial variables; for males only the scores for B-type problems were important. For females three spatial tests shared the prediction of the problem solving composite. For males DAT Space Relations was the most important. Since the coefficients for Card Rotations were almost the same for both sexes, this might be stated another way: for females Cubes Comparison and DAT Space Relations have equal weight, but for males the DAT took the greater share.

Table 4

Canonical Correlation Analysis Spatial Composite with Problem Solutions Composite in Schonberger's 1976 Study

Variables	Standardized Coefficients Girls	Standardized Coefficients Boys
Dependent		
Problem Solutions A	.39	.06
Problem Solutions B	.72	.99
Problem Solutions C	.05	-.03
Independent		
Form Board 2	.08	-.04
DAT Space Relations	.41	.67
Card Rotations	.34	.38
Cubes Comparison	.48	.20
Amount of the Variance in the Dependent Variables Accounted for by this Canonical Correlation	13.31%	7.78%

In summary, there is an indication that spatial and mathematical problem solving abilities are more closely related in females than in males, as was indicated in the studies by Harris and Harris and by Meyer. The connection, at least in the author's sample, is not entirely through the drawing of diagrams, even if the measure is the number of problems for which diagrams were drawn which correctly represented the problem. The amount of both the males' and females' variance predicted by the spatial variables was similar for both solving A-type problems and correctly representing A-type problems with diagrams. However, for the less spatial B- and C-type problems, the spatial tests were more closely related to solving the problems than to drawing correct diagrams for them. Moses (1977) had similar results. For her sample of fifth-grade students spatial ability (measured by Card Rotations, Punched Holes, Figure Rotations, and Cubes Comparison) and mathematical problem solving were more closely related to each other than either was with Degree of Visuality, the diagrams measure.

If drawing diagrams is not the source or at least not the whole source of the connection between visual spatial abilities and mathematical problem solving, then what is the connection? Why were sex-related differences in the relationships indicated in some studies but not in others? One thing that all the studies with sex-related differences in the relationships had in common was that males did better than females on at least one of the spatial tests used. This was true for Card Rotations in Perl's (1979) analysis of the NLSMA data and in the CAA Project's 1970 sample, for Space Relations in their 1971 sample, and for Space Relations in Meyer's 1976 study, with all p-values less than .01. It was for exactly these tests that there were sex-related differences in the correlations with the problem solving measures. In the author's study males' performance exceeded females' performance on Form Board 2 beyond the .01 level. On the other hand, the studies by Fennema and Sherman and by Heseman found no sex-related differences either in performance on the spatial tests or in the relationship between the spatial and problem solving measures. If this were just a matter of male superiority on that which spatial and mathematical problem solving have in common, one would expect the relationship between the two to be closer for males than for females instead of vice versa. Clearly something else is going on.

The tidy classification of spatial tests into the Vz and SR-O subfactors presented at the beginning of this paper was based on the Michael, et al summary of factor analytic studies and on the author's analysis of the topological transformations involved in the tests. It should be noted that most of the studies on which the subdivision was based had all-male samples and the author's analysis presumed that the subject does the spatial items by visualizing the movement. However, there is evidence that there are different methods of solving spatial items and there may be sex-related differences in the choice of method. Barrett (1953) reported different methods for solving items from five spatial tests including DAT Space Relations and three tests similar to Card Rotations. French (1965) found both visual and logical methods of solving Cubes items in an all-male sample. In fact, for the group who solved Cubes items logically the factor structure differed: the Cubes test disappeared from the Space factor and appeared on a factor which appeared to French to measure induction. Werdelin (1961) noted logical methods of solving Cubes items, especially among females. Furthermore, Sherman's (1978) summary of five types of brain lateralization studies indicated greater left-hemisphere involvement in spatial tasks for females despite the usual assumption that spatial function is located in the right hemisphere and verbal-logical function is located in the left hemisphere.

Zimmerman (1954) has suggested that the SR-O and Vz factors represent stages along a continuum of complexity or difficulty and offered some experimental support for his assertion. The author hypothesizes that visualizing nonrigid transformations is harder than rigid ones and that visualizing transformations in three-space is harder than in two-space. At some point of difficulty--which differs for each subject--it becomes easier or necessary to encode the stimulus material verbally and reason logically about the figures. This would explain why solving mathematical problems is more closely related to Vz and three-dimensional tests than to SR-O or two-dimensional tests. If, on certain tests and with certain samples, this happens sooner or more often for females, one might find sex-related differences in performance because the logical methods are slower, and speed is more or less of a factor on all spatial tests. Also some test items may be solvable only by visualizing the transformation; the author suspects that this is true of form board tests and Werdelin's (1961) Sections, tests on which sex-related differences were most evident. This hypothesis would also explain why in samples where there

were sex-related differences in performance on spatial measures there were also indications of sex-related differences in the relationships between spatial and mathematical problem solving.

Introspective or clinical methods would be needed to test this hypothesis directly, but there is other indirect evidence of support for it. There is another type of test which presents figural stimuli to measure what has been called figural or abstract reasoning or cognition of figural relations. (See Figure 6.) Earlier in this review it was stated that in most cases the correlations between spatial tests and mathematical problem solving tests were at least as high as their correlations with other tests. The only consistent exceptions to that generalization were figural reasoning tests. In the CAA Project studies, the related study by Meyer, and the Project Talent study the relationships between figural reasoning tests and the spatial and mathematical problem solving tests were closer than the mathematical and spatial tests were to each other. Dodson's study also found a figural reasoning test to be a better discriminator of the problem solving test than the spatial scales. Another pertinent note is that in a series of aptitude-treatment interaction studies using graphical and analytic modes of teaching quadratic inequalities (Carry, 1968; Webb & Carry, 1975; Eastman & Carry, 1975) the expected interaction did not occur until the authors replaced Paper Folding with a figure matrix test as a measure of spatial visualization.

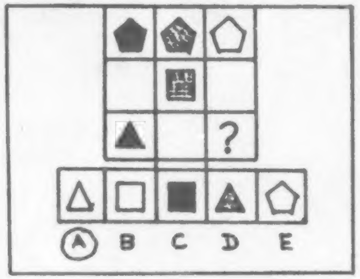

Figure 6. A Sample Item from Guilford's Figure Matrix Test. (Answer: A)

All of this evidence is indirect, and these assertions should be tested directly using experimental or introspective methods. Suppose, however, that it is the figural or abstract reasoning component that spatial tests have in common with solving mathematical problems, and not the visualization of figural transformations. What implications would that have for educational research? First of all, it should caution researchers against such facile conclusions as sex differences in visual spatial ability cause sex differences in mathematical problem solving. Since sex-related differences have not been noted in tests of figural reasoning, it is likely that such differences as do appear on spatial tests are limited to the ability to visually transform the figures. There are honest differences, even among feminist psychologists, as to whether the source of those differences is biological or socio-cultural (Petersen, 1976; Sherman, 1978). However, educators looking for sources of whatever sex-related differences in mathematical problem solving remain in 1979 might better look to sex bias in problem content, differences in mathematics background, or school climate factors such as sex-role expectations than to dig deeper into spatial ability.

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LIMITS AND POSSIBILITIES OF THE METHOD OF THINKING - ALOUD

Our reflections are to be understood as a start in getting an insight into the structures of problem - solving, at least of the problems with we are concerned. Let us begin with some introductory ideas. The most obvious way to explain so - called "thinking" is by considering how one thinks oneself, by observing oneself thinking and then by telling others one's findings. But this method, suitable as it may appear, has its limits. By means of introspection people are not able to be conscious of the processes which take place in themselves in such a way that they can say this or that response has been caused by this or that stimulus. What people can report is, in the last analysis, only a coordination of stimuli and responses which they explain themselves in a way which seems reasonable to them. This is only an image of the actual environment, as is the reality seen by an outside observer. Higher cognitive processes are as closed to the self - observer as they are to the outsider. Thereby natural limits seem to be set which permit a penetration in depth of the structures of thinking only up to a certain point. We don't want to go deeper into this here, but would refer those interested to an essay written by Richard E. Nisbett and Timothy De Camp Wilson of the University of Michigan (1977).

One interesting result achieved by Newell and Simon (1972) is their formalisation of a thinking - problem, the Cryptarithmetic, in which only "thinking - aloud" in the sense of verbal reports is of interest, the subjects reactions in any other form are of no interest. Newell and Simon succeeded in characterising the subject's specefic problem-solving behaviour simply on the basis of the analysis of these thinking - aloud reports in such a way that a computer can reproduce the way the subject reached his solution on the basis of these characterisations.

It is interesting how the tried and trusted method of teaching which relies basically on simply speaking with the pupils, reappears here as the "method of thinking - aloud". To express oneself verbally has become in high technology the basis for the computerised simulation of human thinking. This raises a provocative question: Can we, by using the thinking - aloud of our pupils in a lesson, get a complete enough characterisation of their behaviour in problem - solving to be able to simulate it by means of a computer - programme? This desirable formalisation would be equivalent to a full understanding of the pupil's behaviour in solving problems. Today we are still in the dark in this regard.

A subject's difficulty in expressing himself verbally in a correct way might be an obstruction to the investigation of the problem - solving behaviour particularly of a child on the basis of his verbal expression.

The question is what other - perhaps complementary - possibilities there are for being able to close in on what is happening in a person who is thinking. What actual perceptible effects does thinking have (visible effects, audible effects, etc.)? We need to look for "thinking - aloud" still in the proper sense of the word, but on a different level. What kind of tasks are useful in this regard?

Our concept of the process of problem - solving is to be understood not so much from our exploration of problem - solving behaviour by means of simple tasks (whose treatment consists in extreme cases in giving "yes" or "no" as the answer to a question), but from our choice, like Newell and Simon among others, of more complex tasks; namely tasks which themselves have the character of a process. In the search for a convenient medium in mathematics, the solution of equations or other problems dealing with formal elements are obvious. If an equation isn't of the sort whose solution can be found at once, the sequence of the separate steps used in finding the solution (according to the application of certain rules) can make clear the process used. These steps can in fact be written down, and interpreted as another form of thinking - aloud. This will be made plain by an example:

1) $4 \cdot x = 2 \cdot (x + 1)$
2) $4 \cdot x = 2 \cdot x + 2$
3) $2 \cdot x = 2$
4) $x = 1$

To go from the first to the second line, the brackets are expanded, then "2x" is subtracted from both parts of the equation and finally the whole equation is divided by "two". As long as two or even more steps are not made at the same time such a sequence describes clearly the way of solution. What is not identifiable is the deeper reason for the specific carrying - out of the individual steps; why in each case was not another possible operation prefered instead of them? In our example it isn't clear why the subject has begun by expanding the brackets and not first by dividing the equation by 2. Here the method of thinking - aloud as verbalisation could be useful. The subject must express himself verbally during his considerations and the experimenter will question him about certain points.

A special problem exists if nothing clearly rational is happening. To conclude that the subject can't think is without doubt too hasty. Perhaps he has not enough technical knowledge to solve an equation or perhaps he has simply made an oversight, or perhaps he really has made some mistakes of thinking. Perhaps the fact that the task and the method of its solution are on the symbolic - level is an emotional barrier. - What must we interpret if he is scratching his head or tracing the elements of the equation with his pencil or perhaps has come to a stand still on a certain point? His handling of the equation remains obscure. The equation is clearly of a sort which is too inflexible, too fixed for our purposes. It does not offer enough possibilities to elicit observable responses from the subject.

Because of these, and for other reasons not mentioned here, the results of experiments could be falsified so substantially that no trustworthy statement about problem - solving is possible.

Conclusions drawn by us from these considerations are to be found in our experimental procedure which I'll now try to describe.

In our experiments, children work with building blocks (Dynamical Mazes) (see: Cohors-Fresenborg, 1978). "Work" here is to be understood in the same way as it is in other hard work for the children: e. g. baking cakes with sand; it is a game, though it is taken seriously by the children.

The children are set the task of sticking the bricks onto an appropriate board to construct automata like a chewing - gum - vending machine, a bottle - sorter, an addition - machine... More properly speaking they are constructing the switching-part of these automata.

An automaton in the wide sense of something which functions automatically is an object of fascination for everyone, and particularly for children who naturally like to explore as for example when they take their playthings to pieces.

An important aspect which must be kept in mind when working towards the kernel of the experiment is, how to form a sort of introductory orientation phase which gives the subject time to make a break from the other cares of the day to be able to concentrate himself on the task. The introductory phase gives him a feeling of security that he knows what he is doing. He develops confidence in himself and in the experimenter. If this is not reached the reactions which can be observed cannot be taken to reflect what is really happening inside the subject.

Therefore in our experiments we begin very simply with the presentation of the bricks which they have not seen before (the same starting-situation for all). We ask the children to find suitable designations for the bricks. This presents them with no difficulty. It is the childrens' imagination which is addressed. By means of this kind of introduction to the task natural communication is retained during the whole experiment. One must be conscious of the danger of the extent to which an experiment so determined by the introductory phase is contingent upon the personality of whoever conducts the experiment. How does he succeed in communicating with the children at a level which they can understand and repond to?

The bricks each consists of at least one indentation, the entrance, and one bulge, the so-called "nose", the exit, which are joined together by a groove. The child fixes the bricks on the board, "noses" being fitted into appropriate indentations. With a small rod the children can follow the grooves through the bricks and so test the functioning of the automaton.

The driving-rule is: follow your (the) nose! But it is forbidden to make a left or right turn.With the bricks the children can work in a concrete way. They can take them in their hands, test what they construct and first build parts of an automaton before solving the whole problem.

After the section which usually takes less than five minutes the child is given a problem which he has to solve.

These tasks are sequential on two levels: on the one hand the child is required to construct the switching-part of a machine which he must develop brick by brick, on the board and on the other hand, inductive problem-solving is required, e.g. a two-bottle-sorter has been built one knew that a certain element was a stumbling block, but not why, here it is far more meaningful when children in driving through the net stop at a point; for example if they notice that the automaton has not switched over correctly. One can see what the children are doing, when, where and by cause of what they are moving from one step to another. One can see if they take a brick with apparent certainty and yet hold it in their hands for a certain time or fix it at once at a suitable place. How do they find their solution, do they exhibit trial and error behaviour, do they really construe the automaton as a functioning machine or do they seem first to prepare for themselves a plan, perhaps of the mathematical idea?

Well,let us follow the building segments of a child on his way for finding his solution. The problem is to construct a two-bottle-sorter. That means to build a machine, which sorts two species bottles arriving on an assembly belt in the orderd succession: apple juice, beer, apple juice, beer, ... (A,B,A,B...) so that all the apple juice bottles come into one box (i.e. these bottles come out of one exit of the machine) and the beer is put into another box (i.e. those bottles come out of a second exit).

The child begins with a brick called points (fig. 1). It has one entrance and two different exits. The question as to where one comes out can be regulated by two adjustable-entrances; one on the right side the other on the left side of the points. (If one drives through the entrance on the left the way to the left exit, exit 1, is free; the working of the right exit, exit 2, is analogous).

Figure 1:

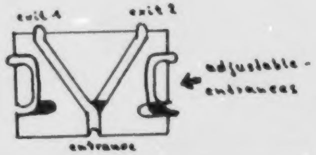
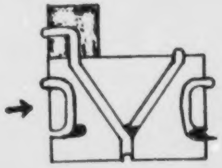


Figure 2:



The child's points are set to the left. What is the child doing? Look at what brick he takes and where this brick will be fixed. It is a left curve and is joined to exit 1 (fig.2). Then the child takes another such brick, one brick called crossing and a third left curve, ... and finishes the way as shown by fig. 3. We know that this method brings no results, because the points yet are set to the left. What does the child will do? We see him taking his rod, driving through the grooves in the bricks and stopping in the adjustable-entrance.

Figure 3:

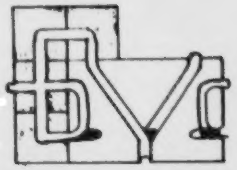
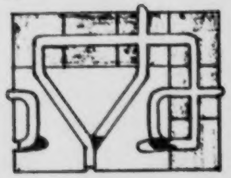


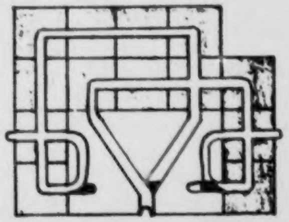
Figure 4:



The child notices that his chosen way is not the right way to transpose the points. We watch him removing all the bricks except the points and starting at the same exit with a right curve. Now he builds a route to the other adjustable-entrance (fig. 4).

When he has finished it he again takes the rod and sees that he has been succesful. Then he constructs brick by brick an analogous path from the right exit to the left adjustable-entrance. With the rod he tests his automaton and seems to be proud of his work. (fig. 5)

Figure 5:



All this can be observed and explained simply by means of the pictures in a much more exact way than I have described the process here. What is important is that no verbal expression of the child is needed to understand what he is doing, to follow his problem-solving way. This method seems to be an interesting possibility for exploring the problem-solving behaviour even of younger children who are, in fact, only able to speak of these and those "things"... which cannot possibly be used to explain their thinking.

What about the thinking processes of deaf-mutes? It may be a possibility to investigate this by means of the bricks.

But it will be shown that these pictures in certain cases are not sufficient. Therefore we need our third kind of record: the audio-visual report. The child is asked now to consider a 2 x 10 Pfg. - chewing-gum-vending-machine. This automaton is the same as he has just constructed but with a different interpretation of the function of the and now the problem is to construct a three-bottle-sorter and to generalize to an automaton with n states.

In conducting our experiments we are exploring the childrens' behaviour in algorithmic problem-solving. This can be seen as sequential thinking, a direct and specialised form of problem-solving behaviour in general.

It has proved to be advantageous that the children work with the building-box. Because of this we can rely on another form of thinking-aloud: the net of an automaton constructed brick by brick. In contrast to the Cryptarithmetic where the main volume of the information which brings understanding is carried by verbal expressions leading to a written report based on these verbal expressions, we have a sound-film-report. This gives us the possibility of working with three sorts of reports and of comparing our conclusions from each with the other:

First: to turn off the pictures and to have only the sound-track. Audio-report.

Second: to turn off the sound-track and to see only the pictures. Visual-report.

Third: to work with both sound and pictures to draw conclusions about the process of thinking. Audio-visual-report.

It has been shown that from the sound-track alone without the appropriate pictures the way in which the child has reached his solution is not at all clear. There are so many difficulties with the spoken word. Do the words the children have used mean what we take them to mean? Did they perhaps want to say something else with a different emphasis? Basically one can say that in what is for them new and abstract terminology the children have problems in expressing themselves, although to have the machine as a model for more complicated algorithmic relations facilitates their expressions. Like someone who is not a good master of a language they make complicated transcriptions which may falsify the proper sense; they speak about "this thing", "that thing", "if I put this here, or that there..." etc. and never formulate whole correct sentences. They don't say e.g.: "Now I'm building a right-hand curve at the first exit."

Let us now turn to the visual-track. The task and the way it must be treated are such that simply by means of the pictures many parts of the problem-solving process are obvious. Brick by brick an automaton takes shape, each step of change is represented by an additional brick which is fixed or one which is removed. Unlike in the equation when exits (and states): exit 1: more money; exit 2: chewing-gum is given. What does the child do? We see him taking two straight-bricks and fixing them together on the board. Then the child drives with the rod up and down through these bricks. We cannot understand what he is doing, or explain what he is thinking. With the sound we come to know that his idea is the following: One coin simply arrives at the middle of the bricks and the second pushes it to the end, the exit of the machine, so that the chewing-gum is given.

Take again the problem of constructing the bottle-sorting-machine. The child begins with the points and continues, as it seems to the observer, with random trials. Is he stupid? The sound-track provides us with information about the child's way of thought. In his mind the bottles are too close to each other on the assembly belt so that one bottle cannot switch the points before the next one comes into the points. What we see are his attempts to arrange this.

So in the cases where the pictures alone are not enough to understand the thinking of the children we need the sound. Their difficulties in expressing themselves verbally are of no importance here because by pointing to the net with their fingers the children are really able to explain very well what they mean.

In broad outline, what I have described is an extract from a pilot study carried out by our research group at Osnabrück in Feb./March this year. The results have been such as to encourage us to continue the research to cover far greater number of children.

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A Child's Eye View Of Learning Mathematics. Has Piaget Found It?

Mary Sime.

From at least as far back as classical times great educators, working intuitively and pragmatically, have discovered viable and rewarding teaching methods and have propogated them without probing analytically into the reasons for their effectiveness. In contrast it took a scientist of our time, rather than an educator, to be the first to give us an analysis of how we learn, leaving us to decide how to apply it in teaching. I say this because Piaget's research has stemmed essentially from biology and so, for him, epistemology is essentially a branch of human biology. Fortunately for us he is, at heart, a mathematician too, so his diagnostic tests of children's learning delve particularly deeply into the growth of mathematical primary concepts in children and into the mathematical forms, or secondary concepts, that thinking can take on in adolescence.

I have based most of my own work, over the last 20 years or so, on observing such self-motivated learning in children and on helping students to devise teaching methods that are in harmony with such learning. During the last two or three years I have been conscious of a satisfying synchronisation of Piaget's theories with the aparently contrasting beliefs of the linguistic philosopher, Prof. Noam Chomsky. I speak of contrast because, whereas Piaget's theme is that all valid learning grows from interaction with the environment, Chomsky claims propensities towards language and mathematics are pre-programmed in the human race. If one keeps both these hypotheses in mind it is rewarding to experiment in ways of offering children heuristic learning that they themselves will spontaneously exploit to the full.

As one fluctuates between Piaget's and Chomsky's theories one soon finds that, either way, conceptualisation of any field within mathematics means much more than mere understanding. It means that understanding has to continue to deepen until it emerges as an indestructible, dynamic part of one's self which reacts on one's behalf as impulsively and inevitably as one's limbs or gastric juices do.

This precipitates at least three questions:-

1. How can we ensure a strong mathematical element in such conceptual growth? - whether or not an outline structure is pre-programmed.

2. If it is pre-programmed are there "critical periods" in its growth? We accept that there are "critical periods" in pre-programmed physical growth: we accept that such physically-focussed "critical periods", (e.g.crawling)can influence mental development for good or ill. Are there similar intellectual periods in which mathematics enters a new phase, with new

mathematical skills to be either nurtured or lost for all time? Even if mathematics is not pre-programmed are there points in the recognised Piagetian intellectual periods at which particular mathematical propensities should develop through meaningful interaction with the environment, but yet could be destroyed at the very moment of emergence?

Pre-programming or no! - why is it that most pre-school children respond vigorously and enjoyably to mathematical properties in the environment but most adults, however able they are in other subject areas, claim to be un-mathematical?

3. How can we best teach students to study children as well as studying mathematics, so that they in their turn can answer the above questions and apply their answers to children?

To search for partial answers it is easiest to look at the learning of mathematics in each of the five main intellectual periods.

I. The Sensory-Motor Period. (Babyhood).

No matter whether it is due to pre -programming or to an innate urge to interact with the environment, the average baby absorbs plentiful pre-mathematical learning. Every environment is rich in mathematical content. Every parent can, if she has the astuteness to do so, increase that learning by adding to the mathematical properties in his playthings and by joining in the fun he has in his explorations. She is, after all, the most important feature in his environment.

Piaget claims that such learning starts in the first hours of life. Mathematical properties such as direction and proximity are studied through the senses as soon as ever a baby searches for his first feed. The searching for food is certainly pre-programmed. Is the *interpretation* of direction and proximity equally pre-programmed? Either way, the quick abandoning of non-rewarding directions of search would suggest discrimination and learning.

As weeks and months go by all sensory experiences, especially visual and tactile ones, bring to the brain messages that become stamped into the mind and body by"circular reactions" such as repetitive clutching or, later, repetitive throwing.

In "The Origin of Intelligence" we read of perhaps hundreds of such examples of learning as they increase steadily in sophistication across the six sub-periods of babyhood. If one probes deeply into each example one finds some element of pre-mathematical worth in perhaps half of them. I prefer to call it "pre-mathematical" because it is still tied to the sensual: the abstractness of mathematical thought is missing. Generally in these examples the baby is exploring the physics of his world. As he reaches the stage of crawling and then of walking his examinations bring him even more pre-mathematics to assimilate and accommodate to. By clutching and gnawing whatever he can reach he begins appreciating shapes, sizes, flexibilities, weights.

By the time he approaches the preconceptual period (at some time about 20 or 24 months) other ideas that have mathematical form begin to take hazy shape in his mind. For the sake of time I will leap over all but one more babyhood example.

Piaget tells us of his daughter, Jacqueline[*1], at 18 months old, whose ball had run under a sofa and out of sight behind it. Jacqueline turned her back on the spot where she last saw the ball as she ran round obstructing furniture to retrieve it. Piaget recognises in this action an intuitive dawning in Jacqueline's mind of thinking in the closed form of a mathematical group. I presume Chomsky would see it as a pre-programmed mental skill. In my work I have come across many such examples, some of which involved not only running round but also climbing over (for the baby's pre-geometry is three-dimensional and not flat).

Little need be said about adapting teaching techniques to serve a baby's learning. The parent who plays and talks with her child enriches his learning enormously. The only question to be posed is of how the parent can be made conscious of the wealth of properties of mathematical worth that should come into such play and conversations.

II. The Preconceptual Period. (Toddlerhood).

With the onset of the superabundant energy of toddlerhood the child's explorations and his consequent global learning become more vigorous and easily observed. In the three years of this period we can watch preconcepts gradually emerge as a result of exploratory experiences that are usually based on play and are culled over in the mind in moments of seemingly idle meditation.

Any one preconcept is the essence left in the child's mind and body from a medly of experiences that have a common feature. Preconcepts are vague and insubstantial: they inter-tangle with one another. They are dynamic and so motivate him to further activity. Frequently towards the end of this period they combine in his thinking and lead him to illogical[*2] conclusions. For example, Paul, a 2-year-old, was"mending" his tractor by twisting his screwdriver through a hole comercially designed for other purposes. He explained that his mother mended things that had holes and his father used a screwdriver to mend the car. (Incidentally, his tractor's breakdown was due to a jammed wheel.)

Many of the preconcepts are of a pre-mathematical or pre-logical nature. They may be none too obvious to the parent if she herself is less alert in mathematics than in other fields. The child is still drawn most frequently to vividly visual and tactile playthings. He lays them in non-logical patterns, mixes them, measures them, shares them out round the family. Consequently he begins to fabricate such ideas as "enoughness",

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[*2] I treat logic and mathematics as inter-related.

"fullness","nearly-fullness", "sameness", "more", "less". Every one of these ideas, maturing gradually over the period, is absorbed into him to a point at which we call it a preconcept. Perhaps it can evolve into more than one. "Enoughness", for example, can branch out into preconcepts of area, mass, length, number, addition, subtraction and "shares for all" (division). "Sameness" may be the first small seed of what will, in many years to come, produce the algebraic equation. Geometrical preconcepts emerge from manipulating the variety of shapes in kitchen junk and in comercially designed toys. Topological preconcepts come from playing with string or with the garden hose or from trying to tie his own shoelaces. Pre-logical classification comes from "helping" mother lay the table and sort out knives from forks and from spoons after washing and drying them. With her he builds up experiences of small numbers in similar fashion. Preconceptually he meets the applied mathematics of physics as he plays on his swing or sea-saw or as he tries to sink his wooden boat.

Now, as in period I, there is little point in talking of teaching mathematics. Parental encouragement to learn it is supremely important. Piaget makes much of the concurrent onrush of speech development that helps a toddler to clarify his thinking. Chomsky implies something similar in seeing their programmes as interrelated. Every mother knows that her child tries to talk to her about everything that excites him. So, at least, this must be an ideal time to lay the first habit of talking about his pre-mathematical findings. (One of the sadnesses I find in school and adult life is that few people's conversations show an enjoyment of mathematical features. Is this a taboo, built in babyhood?)

A mother's patient discussions about her child's own discoveries will help build in him concurrent, interrelated growth in language and mathematics only if she recognises the potential mathematical worth in particular activities; otherwise she may even steer his activities and thoughts away from mathematical experiences of which *she* does not appreciate the worth and towards experiences that interest her more than him. Yet many a parent, anxious for her child to shine intellectually, tries to force early abstract mathematics into him before his brain is ready to receive it, causing him stress and blunting his ability to learn. This may cause a bottleneck in his mathematics that many a teacher could not clear.

Let two examples substantiate this point.

Briony had been taught to recite numbers. Questioned by her father about five bouncing dachshund puppies they met, on a multiple lead, she squealed excitedly, "One, two, three , four, five, six, seven. I counted them."

James, in contrast, whom I met in the village street, was hopping from object to object, counting his hops aloud to his mother. Not only was he experiencing numbers in a very real sense: he could estimate in advance the number of hops needed for each venture.

III. The Intuitive Period. (Infancy.)

[In looking at the intuitive period some of us who share a seemingly common language need to beware of the wide divergences in meaning of the word "infant". In England it means a child of between the ages of about 5 and 7 years; at 5 he starts compulsory schooling. In U.S.A. it means someone slightly younger, and he starts school at six. He is what the Australian so charmingly calls the "kindy child": there, first schooling varies from state to state. I intend in this paper to use the word to mean any child in the intuitive period.]

This is the earliest period for which I have trained students to become teachers. Unfortunately it is also the one for which the largest number of non-mathematical students apply to teach. Yet, from the child's point of view, it carries a whole series of developments which have a vital impact on the growth of pre-logic and pre-mathematics.

It is the period during which most primary preconcepts gradually strengthen into primary concepts. This move forwards in mental maturity gives the child a sense of security and he becomes ready to socialise spontaneously with others of his age (as contrasted to playing parallel with them) and to take an interest in their thoughts and conversations rather than only in their toys. Hence he can, without much trauma, be amongst numbers of children and still learn as intensively as he would at home. He still learns predominantly through play but now he is also interested in learning for its own sake.

If one examines his mathematical thinking with the recognised Piagetian diagnostic tests one finds that, during infancy, he is passing through the middle stage in most primary concepts. Towards the end of the period he shows a fairly good understanding of simple examples within conceptual themes but it is only at the very end that understanding grows so deeply into him that it becomes a dynamic part of his mind and of his whole biological being. Then the relevant primary concepts can be said to be safely formed. From that time on he can neither resist nor intentionally destroy them. They are part of him.

If he is fortunate in his teacher many of the simple concepts that he forms will be mathematical ones such as length, area, seriation, classification, number, topological closure, curvature and straightness. Since he is still very responsive to adult suggestions he will work keenly at a task that an adult sets him: he will struggle, alone or with one or two friends, to exercise his preconcepts on materials relevant to that task. The teacher-in-training needs to *foresee* the relevance of what she suggests and supplies. She needs, at frequent intervals, to discuss findings with each nucleus of children, not only to help them with their thinking and to create a habit of discussing mathematical ideas, but also to register the progress of each child towards conceptualisation.

All too often I found that most of my students tried almost or quite to ignore the provision of a mathematical element in general conceptual growth. Consequently mathematical pre-concepts became dull and withered. Other students, contentedly, left children alone to manipulate Cuisenaire rods or to play with various materials that the teacher had supplied, showing little or no interest in pre-mathematical achievements nor suggesting further lines of mathematical adventure. They would, for example, leave children to tile the floor of the Wendy House (while themselves concentrating on other children making wooly lambs). Never would they, of their own accord, think of discussing the repetitive geometrical patterns that could evolve from the variously coloured tiles, nor take advantage of the number of rows and of tiles in each row. The shapes of rectangles, triangles, squares and hexagons were never exploited. They would leave children to put elastic bands on pin-boards without holding relevant conversations at each table. They brought no mathematical imaginativeness to the general ventures. In spring many a classroom has beans growing in jars on window-sills: such students would encourage this and make the room look gay. There the learning ended. They would not foresee the chance of covering the earliest roots with a removable cloth (to encourage roots to grow near the glass) so that they could use further root growth for counting, measuring and graphing exercises. In short, they considered the mathematics of the intuitive period needed to be nothing more than juggling with numbers.

I think the most crucial words I have used are "without foreseeing". Such lethargy towards venturesomeness in pre-mathematics worried me. It was not sufficient that, if I visited again next day, every child was counting squares on pin-boards. That was far from implanting in him a lifelong habit of mathematical quest. It became obvious to me that every student needs to be led to see the child's point of view and to appreciate the child's eagerness and ability to revel in pre-mathematics. He also needs to learn some simple mathematics himself.

I will describe my approach to this problem when I have pin-pointed similar difficulties and needs that arise in the next intellectual period.

IV. The Concrete-operational Period. (Junior days and early adolescence.)

If there is such a thing as a "critical period" in the learning of mathematics I would think this is it. Once juniors have the motivating power of mathematical concepts built into themselves they are impelled to manipulate any materials that come their way and to exercise their fast-developping reasoning powers in the process. Boats, houses, spaceships are built with meticulous care and with some attention to size, proportion and viability. There is a certain satisfaction in reasoning out loud to one's peers and weighing their arguments and counterbalancing them with one's own. So, gradually, gangs of like-minded companions are formed: within these gangs projects are hatched and carried out. What a wonderful fund of energy and enthusiasm for a far-sighted teacher of mathematics to harness! What an invitation to encourage mathematics within heuristic learning!

Nevertheless about 15 years ago I still found my students ignoring this opportunity. So-called projects were fashionable at the time: they rarely contained learning in any rigorous subjects. For example a student would take a class of children to Preston docks, have them shown round storehouses of wheat and bananas, encourage them to write down un-imaginable numbers of tons imported from various countries, go back, write an essay (often loaded only with incidental happenings such as the bus being late), paint a picture and make a guesswork model of the docks completely lacking in shape and proportion. They often walked un-seeingly past all the mathematics displayed in the cranes and past container lorries ergonomically designed for a variety of specific loads. They took no measurements. They did no reasoning. There was no mathematics in their visit or in its follow-up work.

Such students, in the classroom, rely heavily on textbooks for mathematical work. Some of the textbooks are, of course, quite excellent. But they can be used either imaginatively or dully. They are not meant to be used full-time by a child working solo, encouraged only by ticks from a passing teacher. The junior child still needs to discuss with his friends and his teachers his successes as much as his difficulties with the tasks that a book sets him. He needs such shared interests to keep his mathematics challenging and enjoyable. And, equally, he needs good project work to add zest to it and to keep his mathematics alive in a real-life setting.

The researches of Piaget show that, in this period, the child's reasoning is nurturing the embryonic growth of what will be the secondary concepts of late adolescence. Then they will be the thinking patterns, as for example of reciprocal implication or of holding variables constant, with which he will reason completely abstractly: I will deal with them in the next period. In this period he needs to experience the essence of these future thinking patterns but he is only able to get such experience in practical settings: the challenging conversations help him to interiorise the experiences. Within his gang he will also develop and practise aptitudes such as recognition of accuracy or inaccuracy, respect for accuracy, recognition of contradictions (first of his peers and then of his own) and he will begin to value the logic within classification. He begins, intuitively, to try to sort out negative factors in a problem but is erratic and unsure about it and heavily dependent on materials to hold his thoughts steady.

Earlier in this paper I used the words "so-called projects" for classroom activities that do not nurture the growth of rigorous subjects. Nevertheless I am convinced that genuine project work, rich in such content, is, like good text-book work, invaluable. They substantiate each other. My students tell me that it is also a way of handling large classes easily. The bulk of the teacher's work is advance thought and advance preparation. In the classroom the teacher tends to be treated as an umpire and advisor. Results and problems about which he is consulted have been thoroughly examined within the gang before they are brought to him and there is an overall atmosphere of co-operation.

Primary Teacher Training With The Above theories in Mind.

The thoughts that I have expressed so far led me to consider that the problem I was facing with my primary teachers-in-training was not merely how to ensure good teaching (by remote control!) for any child "suffering" teaching practice. The major challenge was one of convincing all students:-

1. of the concepts in pre-mathematics that an infant can form and what reasoning standard a junior is able to achieve.
2. that they themselves must stretch the average child to achieve them.
3. that they must acquire the skill of diagnostically monitoring conceptual growth.
4. that they must learn to integrate diagnostic testing with the planning and preparing for relevant, active mathematical content in all primary education.

This could be telescoped into stating that the need was to ensure that the student himself should form fully active concepts both of the essential mathematics and of Piagetian theory.

There were, of course, obstacles in the way:-

1. One needs full adult concepts in mathematics to perceive the foundations of them that are laid in primary life, but the majority of "primary" student-teachers were weak in mathematics and undervalued it.
2. Of the minority that were mathematically able, (a) a few were not only skilled but enjoyed mathematics, (b) many were mechanically skilled but did not enjoy it.
3. Even a good mathematician cannot imagine being unmathematical.
4. Even the best of all students is harrassed by a child's slow thinking and grows anxious to convince him verbally.

Although no abstract thinker can prevent his thoughts leaping ahead of concrete reasoning he can, intellectually, learn to understand the child's point of view. So, for about 15 years I subjected all my first-year students (all were "mature") to a weekly afternoon of Piagetian theory and practice throughout the autumn term. I lectured with transparencies. In partnerships students made a wide variety of Piagetian apparatus relevant to the age-range that each hoped to teach. They tested each other on it, often getting surprising results. Then, at half term, they brought their own children in hordes into my enormous lecture room: the children of college staff (academic, domestic and custodian) came too, to "have fun doing puzzles". No-one might mention the word "test" in the children's presence.

The room was always a hive of happy industry all morning, with every child trying to exploit every "puzzle". After lunch most returned for more. A small clientele of children came from year to year so I kept records of development.

Of course, on the first occasion the students, who had blandly assumed that diagnosting testing would be easy, met the difficulty of follow-up questioning to probe the deeper thoughts of each child. Some apparatus was too flimsy and it broke. Some students simply found that their own thoughts were more muddled than any child's: they had not fully understood, or in many cases were unable to believe, what I had tried to teach them and their reading had been superficial.

In discussion groups we analysed responses that had been taped or filmed (by C.C.Tv.) or that students had noted in detail. A few weeks later a headmaster brought in an age-range of about 20 children, now all primary, for a further afternoon of testing. At this second attempt students had far fewer casualties but they were still far from perfect.

By end of term students had been motivated sufficiently to:-

1. accept a great deal of Piagetian theory work.
2. sign for a short course in primary mathematics for anyone who needed it.
3. test almost any child who visited their homes or whom neighbours would lend them and to watch for evidence of conceptual growth in children's spontaneous activities.

At the beginning of the spring term I generally felt secure in asking headmasters to accept partnerships of students into a classroom, for a weekly afternoon, to carry out both diagnostic testing and a simple experiential learning project, rich in language and mathematics and based on their test results. In some years I substituted for this four quite ambitious concurrent group-practices (six students in each class of children) for a day a week, in primary schools geographically close enough to each other for me to pay several visits to each one each day.

In time for teaching practice in the first summer term almost every student had prepared practical work and materials for primary children's experiential work and could give me advance estimates of the consequent learning at least in language and mathematics. Most over-estimated. Always some students did a really good practice. The majority had some disappointments mixed with at least equal successes. Often there were a few who underprepared and who bored the children with long spells of empty activity, with un-discussed text-book work and even with "talk and chalk".

V. The Period of Formal Reasoning and Hypothetical Thinking.

(Late adolescence and adulthood).

The young person who has reached the period of formal reasoning is one in whom secondary concepts (essential to mathematical thought) have matured. He will apply them irrespective of the mathematics syllabus that he is following. It is the secondary concepts rather than the syllabuses that concern me in this paper.

The transitional period into formal thinking is a long one. During this transition an insecure, erratic and frequently imperfect attempt at formal thinking is struggling in the young person. In brief moments it can give the adolescent a feeling of power and elation which can hurry him into inaccuracies and incomplete reasoning, leading to unwise decisions. Because the growth of secondary concepts normally coincides with glandular growth, frequent failures in reasoning can produce depression which could result in fear or hatred of mathematics and a withdrawal towards less rigorous subjects.

Secondary Concepts:

According to Piaget secondary concepts are models of complicated mental activity that have mathematical form and dynamic force. This is not an analogy. He considers, and has convinced me, that they become well-trodden paths in the human nervous system and brain and are as programmed as they might be in a computer. But a secondary concept is richer than a computer programme for, not only does it solve problems but it is also self-activating: it asks questions and wonders beyond its solutions. Secondary concepts may act at lightening speed: at other times we apply them consciously and laborously.

The mathematical forms that secondary concepts can take are perhaps innumerable. The simplest ones, such as concepts of proportion, reciprocal implication and holding constant of variables, I mentioned in their stage of preconceptual growth in the concrete operational period. Most of them are "closed" in group form. Before maturity Piaget calls them groupings. When they mature, i.e. become completely abstract, they enable one to hypothesise beyond all evidence and to construct formulae for testing hypotheses. Hence the word "formal" thinking! I presume Chomsky would consider them pre-programmed: Piaget sees them as acquired through experience.

The secondary concept or mental model central to Piaget's work is that in the form of the Klein 4 group. His best illustration of it is of mental reasoning about a snail moving to and fro over a board which moves to and fro over a fixed spot. In his more publicised example (less convincing since its elements rarely all act concurrently) he follows the thinking of children of all ages in manipulating the movement of weights on the arms of a balance[2]. The element of negation could be the thought that prompts one to take off a newly added weight. One might compensate with a weight on the other arm. These are two simple groupings that quite a young child would discover. Correlation of the two would only be discovered mentally by hypothesising: then one could test the hypothesis mentally or pragmatically.

When a person can hypothesise and carry through the proof or dis-proof, thus creating a formula, then his thinking is formal. With use this formula grows into a secondary concept that will respond to a need impulsively. It will eliminate concrete reasoning because of its greater speed of action. Meanwhile even an infant can understand, and even be drilled into obeying, this particular formula.

Secondary Teacher-Training With These Theories in Mind.

Here the task was more complicated than for training primary teachers:-

1. Most children in the first year of English secondary schools are still concrete-operational as also many will be in the classes above them.

2. Many more, mostly in mid-adolescence, will be passing through the long, critical intermediate period and are still dependent on plentiful pragmatic learning. A teacher needs to give them sensitive and patient help.

3. The formal thinkers need the most alert and skilled teaching of all if they are to be stretched to the limit of their potential. One must never let their mathematics become dull and mechanical. One must challenge them, discuss with them often at a high level of mathematics, discuss and appreciate their thoughts and their hypotheses, urge them forward beyond each

success as it comes. One needs to pass over to them one's own enthusiasm as well as one's mathematical knowledge.

Most teachers of mathematics in secondary schools will, of course, be mathematics specialists and therefore nimble thinkers. They should find it easy to stretch the most able. For the same reason they should be the most able at understanding Piaget's theories and applying them to the wide range of abilities.

With this need for great versatility of teaching method in mind I imposed a first year of Piaget study on secondary students together with the primary ones. They made Piagetian apparatus for diagnostic testing at the boundary line between concrete-operational and formal learning. They joined in the practical sessions of testing and all the follow-up discussions. They did their first teaching practice in a junior school and generally taught sympathetically and well, introducing a strong element of mathematics into the children's general progress.

In the second and third years almost all of them, from choice, continued Piagetian probing, not only of secondary concepts dawning in the adolescents,with whom they now came into contact in the schools, but also into the secondary concepts in themselves. They told me it helped them in their mathematics as well as with their teaching. I still remember the first of them to come to me quietly one day and say, "Please do not tell X.! I never really understood mathematics groups until I studied one of them working in me." Each one of these students built his teaching along his own individual lines so it would be hard to give a general description. Each seemed to strive to be a good teacher of formal mathematics while still remaining a practical teacher of children, enabling them to enjoy exercising their own thinking not only on paper but also in mathematical discussion and in quiet rumination. He strove to know when to use pragmatic work even in the diet of formal thinkers. He brought plenty of practical exploration into the mathematics classes of the younger adolescents and of those that he diagnosed as being less able or, for any reason, less advanced.

For adolescents the backbone of mathematics learning is almost bound to be based on text-book examples. It is tempting for a young teacher to rely solely on them; if he does so the most able children will survive but not be stretched. Yet the young person whose thinking is formal gets real satisfaction from responding quietly to their challenge, tempting the teacher further towards the hidden danger of unmitigated text-book work and mechanical use of formulae. The good teacher will always find a place for at least occasional original work. I know of a minor public school in which boys learned much of their mathematics and science for a year or more in actively building a glider: later they became excellent university entrants. In watching such work, in monitoring the formal thinker as he builds formulae for his own practical purposes, in working with him, the teacher is constantly diagnosing progress as effectively as he would with Piaget tests. The routine mathematics in the other part of the time-table is bound to benefit. The young student-teacher should never fear the spells of concrete reasoning,(and even of just sheer fun with puzzles and problems unrelated to the syllabus) to be wasted time for even the most gifted of formal thinkers if he enjoys them.

Coda.

Good teaching of mathematics in the formal period and in the period that precedes it enriches the child beyond the bounds of the subject itself. It feeds the child's hunger to reason and helps him to know whether and when his reasoning remains within the bounds of truth. It helps him to apply such intellectual integrity to other facets of his life. So, ultimately, it influences his character. A mathematics teacher is educating the whole child and not merely the mathematician within him.

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GOALS OF LEARNING AND QUALITIES OF UNDERSTANDING

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Introduction

The present paper is offered as a contribution to a fruitful interchange of ideas which has taken place during the past few years, at various times and places and between a number of different people. A useful feature is that much of this is now in print, so that the reader can follow up the necessarily brief references I shall make to the views of other participants, and read these at full length and in their original contexts. A concern with the nature of mathematical understanding is of course much older than this, and has involved many besides the present participants. Nevertheless, I think that the present interaction is proving a particularly interesting and fruitful one, and the end product may well prove to have taken our own understanding of mathematical understanding several steps further.

This particular interchange began with an article[1] called "Relational Understanding, and Instrumental Understanding". In this, following a suggestion from my friend Stieg Mellin-Olsen of Bergen, I accepted that the meaning with which I had for many years been using the word "Understanding" was not the only one in current - indeed, widespread - use; and to distinguish between them, I adopted Mellin-Olsen's term "relational understanding" and "instrumental understanding".

By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what in the past I described as "rules without reasons", without realising that for many pupils and their teachers the possession of such a rule, and the ability to use it, was what they meant by "understanding".

One year later, came an article[2] by Victor Byers and Nicholas Herscovics, both of Concordia University, Montreal. This was followed in the very next issue by a rejoinder[3] from John Backhouse, of Oxford. Very briefly, Byers and Herscovics suggested that there were four different kinds of understanding of mathematics: instrumental, relational, intuitive, formal. (I will give their formulations of these later.) Backhouse disagreed; his two main points being that he found the concept of intuitive understanding unnecessary, and that since the term 'formal understanding' meant the same as 'understanding of form' he did not see it as a different kind of understanding.

A valuable discussion of these views took place at a meeting of the British Society for the Psychology of Learning Mathematics at Chelsea College, London, in March, 1978. Backhouse and I were there; unfortunately Byers and Herscovics were not, but I was able to continue the discussion with them later that year. Also there (among many others) were Laurie Buxton and David Tall, who kindly gave me advance copies of their own papers on this subject. These are now also in print [4 & 5]. Buxton accepts the distinction between instrumental and relational understanding, but also discusses what he refers to as four levels of understanding: rote, observational, relational, formal. Tall's paper directs our attention to (among other things) the dynamics of mathematical understanding.

> "In this article the attempt has been made to see different types of understanding and, equally important, non-understanding, as different facets of the dynamic development of the mental processes of the learner."

This dynamic development is necessarily related to the goals of the learner; and if a teacher is involved, to those of the teacher also. In my earlier paper I pointed out some of the possible consequences of a mis-match between these learning goals, where one side conceived the goal in terms of relational understanding. This mis-match can be either way about.

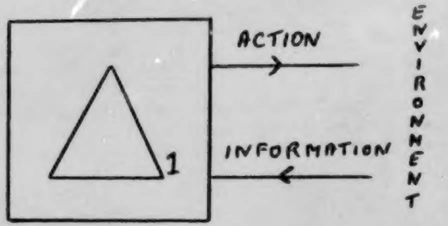
What I have tried to do in the present paper is to bring together what appears to me as some of the main points which have appeared so far from the discussion. Although the various participants have expressed somewhat different views, having re-read their papers more than once, and also played back a tape recording of the discussion mentioned earlier, certain ideas have taken shape which represent a good-sized area of agreement; and I would hope that most of the participants would agree with much of what I am now going to suggest. I am certainly grateful to these contributors for taking up some of my own ideas and developing them in interesting ways; and if I now take up some of their individual suggestions and try to fit them together into a coherent whole, I hope that they will not feel that their ideas have been too greatly modified in the process.

A new model of intelligence

The proposed synthesis has taken place around a new model of intelligence which I offer as a replacement for current ones based on 'I.Q.' All the latter can do is to put individuals approximately into rank order with respect to intelligence. It does not tell us what it is for, how it works, or how we can help learners to make the best use of whatever intelligence they have. In the new model I try to do all these; and, not surprisingly, mathematics appears as an important special case.

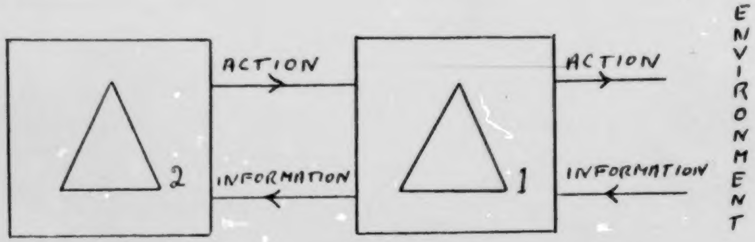
As a preliminary to the synthesis, we therefore need the briefest possible outline of this new model: which is, in essence, a two-level cybernetic one. What follows is no more than a thumbnail sketch - a full exposition takes over 300 printed pages[6].

Our starting point is the observation that much, possibly most, human behaviour is goal-directed; together with the conjecture that cumulatively, success in achieving our goals is a major factor of survival.



(See next page for explanation of this diagram.)

For goal directed activity operating on the physical environment, we have a director system delta-one which receives information about the present state of the operand (what is being acted on), compares this with a goal state, and with the help of a *plan* which it constructs from its available *schemas*, takes the operand from its present state to its goal state and keeps it there. We may if we like call delta-one a sensori-motor system.



Delta-two is another director system, with a difference. Its operands are not in the outside environment, but in delta-one. They are not physical objects, but mental objects. *The function of delta-two is to optimise the functioning of delta-one*. I prefer not to call delta-two a reflective system for reasons which will be given later. In a nutshell: *the job of delta-one is to direct physical actions, of many kinds. The job of delta-two is goal-directed mental activity*, also of many kinds: including learning, but not only learning. Learning includes the construction and testing *by* delta-two *within* delta-one of the schemas and plans which delta-one must have to do its job.

How many kinds of understanding can we usefully distinguish?

With the help of this bare outline, we can make a start with trying to answer the above question, which emerges strongly from the present debate. My original single meaning was expanded to two by Mellin-Olsen, and to four by Byers and Herscovics. Backhouse considers two sufficient. Buxton describes four *levels* of understanding, which raises a new question. Are we talking about different points or regions along a continuum, or about qualitative differences? He calls these levels 1. Rote, 2. Observational, 3. Insightful, 4. Formal. The first three he relates to the instrumental/relational distinction. Moreover, in the discussion at the B.S.Ps.L.M. meeting already mentioned, there was general agreement that Byers and Herscovics on the one hand, and Backhouse on the other, were using the term "formal" with two importantly different meanings: one related to formal proof, and the other to "form" in the sense "This equation is in the form $y = mx + c$". Buxton's use of the term corresponds to the first of these two meanings; whereas for Backhouse, " ... form is a way of representing a concept ...".

So we now have suggestions for seven different kinds and/or levels of understanding - with a tendency to proliferate, due to the creativity and individuality of those who are taking part in the discussion. But too many categories may be nearly as unhelpful as too few.

After considering all of these carefully, I think that we need to distinguish three kinds of understanding: Instrumental, Relational, Logical; and two modes of mental activity: Intuitive, Reflective. (Originally I called these "levels" of mental activity, but this was found sometimes to cause confusion with levels of abstraction.) "Logical" is simply a replacement for the first meaning of "formal", so that we know which one we are talking about. I think that nearly all the rich assortment of ideas which we have been given can be fitted into this framework without doing them unjustice. And we do not need to jettison the larger number of categories: they can be retained to give finer distinctions within the major categories, like "sky blue" and "ultramarine" within the category "blue".

I will start with the three different kinds of understanding. With one small change, I agree with Byers' and Herscovics' formulations.

> "Instrumental understanding is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
>
> Relational understanding is the ability to deduce specific rules or procedures from more general mathematical relationships.
>
> Formal - (Logical in my table) understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning."

However, as Backhouse points out (my italics), " ... we are unable to observe our pupils' feelings and schemas directly, and look for confirmatory behaviour. *As evidence that* A understands X, we accept the fact that A applies X in situations different (in greater or less degree) from that in which it was learned." So in Byers' and Hercovics' formulation, I would simply like to replace "is" in each case by "is evidenced by". Note also that Backhouse's formulation applies equally well to instrumental and relational understanding. He writes " ... situations different (in greater or less degree) ..." In situations which are not very different from each other, a rule will do. This corresponds to instrumental understanding. But if the situations are quite a lot different from each other, no simple rule will do, and amassing a collection of rules breaks down sooner or later. So it is better to be able, in the words of Byers and Herscovics, " ... to deduce specific rules or procedures from more general mathematical relationships". It is these relationships which are embodied in a relational schema.

What, if any, is the point of trying to make and formulate these distinctions? Are we simply engaged in an academic exercise in the pejorative sense of the word? The word "usefully" appears in the title of this section, and the onus is on me to justify it. I shall argue that these distinctions are not only useful, but essential to us as mathematics educators.

Schemas and the goals of learning

If it is accepted that "To understand something means to assimilate it into an appropriate schema"[7], this implies that we can distinguish as many kinds of understanding as we can distinguish appropriate schemas. What decides whether a schema is appropriate or not? Here are two considerations (there may be others), the first fairly obvious, the second not always so.

(i) It has to be appropriate to the subject matter.
(ii) It has to be appropriate to the task in hand: that is, to the goal to be achieved.

As example of the first, if my son asks me "What is a differential?", I need to know whether the subject matter is maths or motor cars before I can understand his meaning. But the same subject matter may be involved in different tasks, and these may imply important differences within what is superficially the same subject matter. A length of rope may seem a simple enough purchase, but there is more here than meets the eye. If it is for a halyard, I want rope which will not stretch, so that when the sail is hauled up tightly it stays that way; so I should buy, say, pre-stretched terylene. A climber, on the other hand, likes rope with good elasticity, so that in the event of a fall the impulsive tension is spread over a somewhat longer period; so he would be more likely to buy nylon. For a mooring I buy two kinds: a long length of non- floating rope from the buoy down to the holdfast, so that I do not create a danger to navigation; and a short length of floating rope for easy pick-up. For all these purposes, we want rope which does not rot.

So if I ask a salesman for "20 metres of 12 mm rope", he should ask what I want it for, or he may well sell me the wrong kind. He cannot do his job properly unless he knows the different kinds of uses for rope, and the qualities which are appropriate for these uses.

Likewise for mathematics, which we may indeed be trying to "sell" more actively than some of our pupils are wanting to "buy". Although the subject matter (mathematics) may be the same, for various persons and on various occasions the goals of learning may well be different; with the likelihood that different kinds of schema may be appropriate. So if we are trying to answer the question "How many kinds of understanding can we usefully distinguish?" where "we" means ourselves as mathematical educators, then "usefully" now takes the meaning "appropriate to the different kinds of learning goals set by our pupils and students". The distinction between various kinds of learning goal, which may appear superficially alike because involving the same subject matter, is one which we must have clearly available; or we cannot do our own jobs properly, any more than could a rope salesman to whom rope was just rope, without any thought that different persons might be acquiring it for different purposes. We must also remind ourselves that the goals of teachers and pupils may differ. Moreover, if we set up a learning situation which is conducive to one kind of learning goal, this will influence the kind of schemas pupils construct at least as much as, possibly more than, what is overtly being taught. So there will be a mis-match between the situationally-determined learning goals, and those which teachers may have in mind and/or verbally present.

We are now ready to think about three different kinds of learning goal, three associated kinds of schema, and thus three kinds of understanding.

Instrumental Understanding

In schools, the goal of instrumental learning is to be able to give right answers, as many as possible, to questions asked by a teacher (verbally or on paper).

The overt operands (what are being manipulated) are symbols, mathematical and verbal, spoken or on paper. The hidden operand is the teacher, in the first instance. The immediate goals of the pupil are to gain the teacher's approval and avoid the teacher's disapproval. Longer-term, the goal is approval by someone more remote - an examiner, a potential employer.

All of these operands (overt and hidden, short-term and long) are in the physical environment; so the activities described are delta-one activities. The teacher has asked a question and is waiting for an answer; an examiner has set a paper of questions, and the pupils have to answer as many as they can in a given time. The schemas formed by instrumental learning are short-term, the least and most quickly acquirable by which correct answers can be given; in other words rules, which we may regard as degenerate schemas. Pupils learn a set of these, each appropriate to a limited class of tasks. These rules are not entirely isolated - they can be combined, and applied in succession. But the mental structures acquired by instrumental learning have limited adaptability, because these rules are ways of manipulating symbols, and the connections are between symbols and not concepts. Application to substantially new situations requires conceptual connections, which is to say relational schemas.

Relational Understanding

It is the construction of these relational schemas which are the goals of relational learning.

The operands may be newly encountered concepts, and the goal may be connecting these with an appropriate (relational) schema. Achievement of this goal is equivalent to relational understanding, and in the process, the schema itself has undergone further development. Another kind of goal may be to deduce specific methods for particular problems, or specific rules for classes of tasks. Ability to do this is evidence of relational understanding. Yet another kind of goal is to improve the schemas we already have; by reflecting on them to make them more cohesive and better organised, and so more effective for the first and second kind of goal.

In all of these activities, the operands are concepts and schemas within delta-one: so in relational learning, delta-two activity needs to be dominant. And as Buxton has pointed out[8], this delta-two activity can be greatly interfered with if a teacher keeps asking questions! Discussion is quite another matter, but there must also be some quiet time for reflection, and people should not be required to speak if they do not want to. Alternation between these seems to be one of the best ways of encouraging delta-two activity. Discussion gives new material on which to reflect, requires one to formulate one's own ideas as clearly as possible, and shows up weaknesses in our thinking which by reflection we may remedy.

Symbols make an important contribution to all the activities described above. In reflection they act as combined handles and labels for their associated concepts, and we could not have any discussion without them. But the function of symbols is for manipulating and communicating mathematical concepts, and these are the true operands in relational mathematics.

Whereas the pleasure (if any) resulting from instrumental learning derives mainly from pleasing someone else - teacher, examiner - the delta-two activities involved in relational learning are a source of very personal pleasure. We ourselves know that the achievements of relational understanding can be very pleasurable; but many - perhaps a majority - of pupils unfortunately do not know this because they have never experienced it.

The goals of relational learning are long-term, in two ways.

(i) It takes longer to form mathematical concepts and construct relational schemas then it does to learn rules.

(ii) BUT long-term it is the best way of pleasing teachers, passing examinations, satisfying employers, and achieving goals in a variety of future situations for which mathematical methods are necessary.

Relational mathematics thus offers the best of both worlds. Unfortunately many imposed learning situations are not conducive to this kind of learning.

Logical Understanding.[9]

It has been suggested that this category is unnecessary, being no different from relational understanding. But (using Byers' and Herscovics' example) suppose that a pupil writes something like

$$\begin{aligned} x + 3 &= 7 \\ &= 7 - 3 \\ &= 4 \end{aligned}$$

We ask why he writes this, and he answers in a way which convinces us that he has relational understanding. (E.g. "The top line says that adding 3 to x gives us 7, so to find what x is we take the 3 away again.")

If we now point out to him the exact meaning of what he has written, and he then re-writes it correctly, this would be evidence that he was capable of logical understanding, which we had helped to bring about by causing him to reflect. By " ... re-writes it correctly," we mean in an acceptable form which sets down a valid sequence of logical inferences. If, however, he cannot see anything wrong in what he has written, and continues to emphasise that he has given the right answer (as evidenced by putting 4 for x in the top line: "4 + 3 = 7, so I've got it right, haven't I?") we might find it hard to convince him that we were not just being awkward. For there is indeed nothing wrong with his relational understanding, which relates to finding the value of x which makes the equation a true statement. What he lacks is logical understanding, which concerns the relationships of implication between the successive statements. We can thus discern two kinds of not-having logical understanding: not-having it, and knowing that one hasn't; and not-having it, and not knowing that one hasn't, because one has not a schema of a kind which would enable one to recognise its presence or absence.

Logical understanding is closely related to the difference between being convinced oneself, for which relational understanding is sufficient, and being able to convince other people. To emphasise this I propose the following formulation.

> Logical understanding is evidenced by the ability to demonstrate that what has been stated follows of logical necessity, by a chain of inferences, from (i) the given premises, together with (ii) suitably chosen items from what is accepted as established mathematical knowledge (axioms and theorems).

In this activity, the goals are not the acquisition of new concepts, nor the construction of new schemas, nor the devising of new methods to solve new problems.

There must already be in existence as operands for the next stage, the goal now being to be sure that the schemas which have been constructed, the solutions which have been devised, are sound and accurate. The methods are what Bruner[10] calls "the analytic apparatus of one's craft": analysis, and the construction of chains of logical reasoning to produce what we call demonstrations or proofs.

The primary operands are thus mathematical ideas which one already both has and understands. I think that we can also identify some secondary operands, namely one's fellow-mathematicians, whose critical evaluation we have to satisfy before whatever contributions we offer will be received into the body of accepted mathematical knowledge. So I would regard a mathematical proof or demonstration as corresponding to a replicable experiment in the natural sciences.

How does the term 'formal' become attached to 'proof' or 'demonstration'? I suggest that a formal demonstration or proof is a method for exposing one's statement to the judgement of one's peers, by putting it in a form in which every implication is clearly shown, every theorem or axiom clearly referred to. 'Form' here refers to the form of presentation, and 'formal' means 'conforming to accepted forms of presentation'.

Most of us will also subject our ideas to self-criticism before making them public, formally or informally; and the construction of a proof which satisfies ourselves gives us confidence to do so. In some cases, all we want is to satisfy ourselves. But I would regard the satisfaction of criticisms and self-criticisms as secondary to the main goal, and indeed as aids to its achievement: the construction of ever more extensive and powerful mathematical knowledge, coherent, and without weaknesses or internal inconsistencies. This is clearly quite an advanced activity; yet as Laurie Buxton has pointed out[4], at one time,

> "proofs of this sort,of a formal nature, were learnt by rote! Thus the most sophisticated level, valuable only to those who have achieved real insight and then recognised the need for proof is linked with the most primitive, thus firmly ensuring that the matter will not be understood!"

One can hardly find a clearer example than this of the need to distinguish between different goals for learning, and the corresponding kinds of understanding which are required.

What kinds of schema are involved in this kind of understanding? Here I am working in my own frontier zone; perhaps others will help to take our knowledge further. It seems to me that the relationships involved are between the truth-values of statements. E.g. "If (it is true that) O is the centre of a circle, and A, B are points on its circumference, then (it must also be true that) $\overline{OA}$, OB are equal in length". "Then", in this context, means that it follows of logical necessity: not empirically, and not because teacher or anyone else decrees it so. The reasoning involved is syllogistic: "All radii of the same circle are equal in length to each other. $\overline{OA}$, $\overline{OB}$ are radii of the same circle. Therefore ...". So the schema involved seems to be one in which the concepts represent classes of statements, and the connections are those of logical implication. It would also seem that its location is in delta-two, since it is by means of this schema that delta-two operates on delta-one to produce schemas within delta-one which are free from inconsistencies. This might explain its inaccessibility to reflection, which is normally an activity in which delta-two becomes conscious of schemas within delta-one.

The Intuitive/Reflective Dimension

From the new model of intelligence we can also derive two modes of mental functioning. These do not relate to different kinds of understanding. On the contrary: they can occur in combination with all three of the kinds of understanding already discussed, as indicated below.

KINDS OF UNDERSTANDING

		Instrumental	Relational	Logical
Modes of Mental Activity	INTUITIVE	I_1	R_1	L_1
	REFLECTIVE	I_2	R_2	L_2

Though what follows may be recognised as a development of an earlier formulation, to save space I will not compare the two. In the present model, as was stated earlier, the job of delta-two is goal-directed mental activity, including learning, but not only learning. "Goal directed" does not imply "consciously goal-directed", either for delta-two or for delta-one. Many goal-directed activities of delta-one may take place unconsciously, such as those by which we maintain our state of balance while riding a bicycle and possibly holding a conversation at the same time. Likewise for delta-two. Much learning takes place unconsciously; and since children learn from birth onwards, in the present model delta-two is assumed present from the beginning. (I conceive it as innate.) But we also know that reflective intelligence develops only gradually throughout childhood and adolescence. By this I mean the ability to make one's own concepts and mental processes objects of conscious attention, perhaps to describe them, perhaps also consciously and deliberately to change them. So what I now suggest is developing during this period is not delta-two itself, but the ability to shift our centre of consciousness from delta-one (the objects of consciousness being in the physical environment) to delta-two (the objects of consciousness being now in delta-one).

In the intuitive mode of mental activity, consciousness is centred in delta-one. In the reflective mode, consciousness is centred in delta-two. 'Intuitive' thus refers to spontaneous processes within delta-one, in which delta-two takes part either not at all, or not consiously. 'Reflective' refers to conscious activity by delta-two on delta-one.

With this formulation and with that of earlier sections, I can now try to justify the six categories in the 3 x 2 table above by offering examples of each.

Category I_1 This corresponds to 'rules without reasons'. Mechanical arithmetic comes into this category; but not the learning of skills to a degree where they can be performed fluently and with little or no thought, provided that when necessary they can be re-connected to an appropriate schema.

Category I_2 This is an interesting one, since until I had set up this model I would have regarded reflective activity as necessarily relational. But consider this example.

Someone can differentiate the function $y = x^3$

by using the rule $y = x^n$

$$\Rightarrow \quad y' = nx^{n-1}$$

He might have been told why some years ago, but he forgot it quite soon afterwards.

Now he is asked to differentiate $y = \frac{1}{x^3}$

His rule no longer applies; but on reflection,

he remembers another rule: $\frac{1}{x^p} = x^{-p}$

By combining these rules, he gets $y = \frac{1}{x^3}$

$$\Rightarrow \quad y = x^{-3}$$

$$\Rightarrow \quad y' = -3x^{-4}$$

He has given the right answer without necessarily having relational understanding of either rule. And what is more, my own view would be that he had tackled the problem quite intelligently, given that all he had to work with was an assortment of rules. So we have an interesting implication: that reflective intelligence can be brought to bear on material in delta-one which is instrumental in quality. I now realise that this is what I have often been doing when, reflecting on mathematics I learnt many years ago, I came to realise that I had never understood it "properly" (i.e. relationally).

Category R_1 I think that this is what Bruner is referring to in the passages cited by Byers and Herscovics[10]:

> "Intuition implies the act of grasping the meaning or significance or structure of a problem without explicit reliance on the analytic apparatus of one's craft. It precedes proof; indeed, it is what the techniques of analysis and proof are designed to test and check."

In terms of the present model, intuitive understanding takes place when input to delta-one is directly assimilated to an appropriate schema, which thereby structures our perception of the problem. But intuition does not by itself imply understanding - the input may have activated inappropriate ideas, and our perception of the problem may on reflection prove to have been faulty. Hence the need for proof, as Bruner points out.

This emphasises the need to distinguish between intuition and insight. As Byers and Herscovics point out, "Many authors identify intuition with the Eurika phenomenon, the flash of sudden insight". But to do this is in my view mistaken. Insight (which I would equate with relational understanding) may indeed take place by an intuitive leap, but it may also be the result of prolonged reflection.

And the result of an intuitive leap may, when tested by the techniques of analysis and proof, turn out not to have been a genuine insight.

Category R_2 Using the calculus example again, let us imagine a student who not only knew the routines for differentiating the given function, and for re-writing $1/x^p$ without the fraction bar by using a negative index, but also had relational understanding of them. He might make an intelligent guess along the same lines as already described, and then proceed to verify this from first principles. For this he would need to relate these two routines to quite extensive mathematical schemas, including the binomial expansion for a negative exponent.

Category L_1 Can logical understanding be identified at an intuitive as well as a reflective level? I believe so. Most children over 10, and many younger, would quickly say that "All dogs have tails. This animal is a dog; therefore, this animal has a tail" is a valid argument, whereas "All dogs have tails. This animal has a tail, therefore this animal is a dog" is not. Asked to say why the latter was false, they might give counter-examples, such as cats, squirrels. These might be taken as evidence of an intuitive awareness that a single counter-example was enough to reveal a false inference, but it might also be argued that all that was demonstrated was empirical knowledge. Stronger evidence for intuitive logic would be similar responses to statements like "All wonks have tails ..." I have not yet however been able to collect any mathematical examples for category L_1; contributions and suggestions will be welcomed.

Category L_2 Understanding in this category would be evidenced by someone who responded to the example just given by drawing the appropriate Venn diagrams for the two cases. A more advanced pair of examples is provided by Russell's famous paradox. (In a certain village, the barber shaves everyone except those who shave themselves. Who shaves the barber?) L_2 understanding of this took a long time to achieve; yet we may assume that most people had an L_1 conviction that there was something wrong, from the start!

These apart, a major part of all mathematics at sixth form and university level comes into category L_2. Wherever an exercise or problem begins "Show that ..." or "Prove that ...", L_2 understanding is needed and must be evidenced in the reply. And it is even more a key feature at research level, where "full mathematical rigour" means the same as a logically unassailable chain of implications.

* * * * *

Elsewhere[11] I have distinguished between a logical and a psychological presentation, and have argued that it is a mistake to think that comprehension is best achieved by a strictly logical presentation. What is needed first is a presentation which is carefully planned psychologically, in order to bring about relational understanding. Until a learner understands what it is that one is trying to convince him about the truth of, then a logical argument is premature and out of place. But when relational understanding has been achieved, I think that there is a place for logical understanding even at quite an elementary level. Acceptance of a need both to give and to receive a logical proof says, by implication: "I don't expect you to believe this on my say-so. I hope to convince you that what I am saying follows of logical necessity from things we both agree about. If you will follow my chain of inferences step by step, I think that you will be convinced by the exercise of your own reason." This further implies that the final authority for accepting or not accepting a statement lies in the mathematics itself: which is something which, in ways appropriate to the level of the learner, I think we should be teaching from early years onwards.

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COGNITIVE ASPECTS OF PROOF, WITH SPECIAL REFERENCE TO THE IRRATIONALITY OF $\sqrt{2}$

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The criteria by which a proof is judged in mathematics seem, on the face of it, to be quite different when considered by sophisticated mathematicians as compared with learners. The sophistrcated mathematician may concern himself with logical structure, mathematical style, the degree of generality, the aesthetic quality of the proof, and so on. The learner may lack the sophistication to appreciate these criteria fully and may concern himself more with the manner in which the proof explains the result and demonstrates why it must be true, based on his current state of development.

A basis for the cognitive development of proof is already available in the psychology of learning in terms of Ausubel's meaningful learning [1] or Skemp's relational understanding [2]. At any stage in development the ideas presented need to be potentially meaningful (in Ausubel's terminology), which may mean, in the short term, presenting proofs in a radically different form from that ultimately desired. In this paper we will see an example of this phenomenon. The initial proof may be more cumbersome, less aesthetically pleasing, yet prove more meaningful to the learner at the particular stage under consideration. Even so, the long term desire for full sophistication must be kept in mind, yielding two complementary but, at times, conflicting, principles:

1. To present the material in a potentially meaningful manner for the learner.
2. To aid the learner in achieving long term sophistication.

These two principles in fact underly the whole of education. They must be carefully balanced; lack of balance in either direction leads to its own particular problems. One may conjecture that psychologists tend to err by overstressing the first principle (for example in interpreting Piaget's theory of stages) whilst mathematicians pay more attention to the second. Over indulgence in the first can lead to stunted mathematical growth whilst the second can lead to less able students having no growth at all.

When these principles are applied to the understanding of proof, the first suggests that proofs should be prepared in a manner which is intended to be meaningful at the time whilst the second suggests that the teacher should remain mindful of the need to develop more subtle and powerful proof styles, choosing the appropriate time to stretch the student's capacity.

The principles also govern the apparently conflicting criteria of the sophisticated mathematician and the learner. The learner may need more attention paid to the first principle in the initial stages, meanwhile the sophisticated mathematician will have progressed so that the first principal implies that he already has enough experience for subtle proof structure to be meaningful. However, it is not out of the question that he may meet a completely new style of proof (for instance the use of first order logical principals in non-standard analysis) where he may find himself in the same position as the learner.

The irrationality of $\sqrt{2}$

In English school sixth forms a standard introduction to proof by contradiction is the following:

Proof C We show that the assumption that a rational p/q exists such that $p^2/q^2 = 2$ leads to a contradiction.

Suppose that $p^2/q^2 = 2$ where p and q are integers with no factor in common, then

$$p^2 = 2q^2.$$

Thus p^2 is even. But if p were odd, then p^2 would be odd, so this means that p must be even, $p = 2r$ where r is an integer. Substituting in $p^2 = 2q^2$, we obtain $4r^2 = 2q^2$, so $2r^2 = q^2$. The same argument shows that q must also be even. So p and q have common factor 2, contradicting the fact that they have no common factor.

This is an aesthetically pleasing proof by contradiction. However the contradiction does not arise by contradicting the given statement:

"$\sqrt{2}$ is not a rational number"

but a different one:

"$\sqrt{2}$ is not a rational number *in lowest terms*"

Learners often feel a sense of emptiness and lack of explanation as to *why* $\sqrt{2}$ is not irrational. There are also linguistic factors which render this proof very special, namely that the English language contains the words "even" and "odd" to describe the property of being divisible by 2 or not divisible by 2, but no other integer has corresponding words for such a divisbility property.

An alternative approach to the problem may be to describe precisely the nature of the square of a rational number and to show that 2 cannot be such a square:

Proof D We will show that if we start with any rational p/q and square it, then the resulting square cannot be 2.

On squaring an integer n, the number of times any prime factor appears in the factorization of n is doubled in the prime factorisation of n^2, so that each prime factor occurs an even number of times in n^2. In the fraction p^2/q^2 we factorise the numerator p^2 and denominator q^2, cancelling common factors where possible. Then each factor either cancels exactly, or we are left with an even number of appearances of that factor in the numerator or the denominator. The fraction p^2/q^2 cannot be simplified to give 2/1 because the latter has an odd number of 2's in the numerator. So the square of a rational p/q is never equal to 2.

According to the evidence which appears in text books, the mathematical community at large prefers proof C to proof D.

A questionnaire asking first year university students which of these they understood on first reading, and which caused confusion, showed no significant difference between the two, but this was highly coloured by the fact that many students had seen proof C before. A second questionnaire presented similar proofs, but with 2 replaced by 5/8, also asking the respondents whether they were familiar with either type of proof. About half were familiar with C and none with D.

Proof C* We will show that the assumption that a rational p/q exists such that $p^2/q^2 = 5/8$ leads to a contradiction.

Suppose that $p^2/q^2 = 5/8$ where p and q are integers with no factor in common. Then $8p^2 = 5q^2$. Thus 5 is a factor of $8p^2$. If 5 did not divide p, then it could not divide $8p^2$, hence we deduce that $5r = p$ where r is an integer. Substituting in the original equation we get $8(5r)^2 = 5q^2$. This simplifies to $40r^2 = q^2$. From this we see that 5 must divide q and, by the same argument as before, it divides q, contradicting the fact that p and q have no common factor.

Proof D* We will show that if we start with any rational p/q and square it, then the result p^2/q^2 cannot be 5/8.

On squaring any integer nm the number of times that any prime factor appears in the factorization of n is doubled in the prime factorization of n , so each prime factor occurs an even number of times in n . (For instance if $n = 12 = 2^2 \times 3$, then $12^2 = 2^4 \times 3^2$.)

In the fraction p^2/q^2, factorize p^2 and q^2 into primes and cancel common factors where possible. Each factor will either cancel exactly or we are left with an even number of appearances of that factor in the numerator or denominator of the fraction. The fraction p^2/q^2 can never be simplified to give 5/8, for the latter is $5/2^3$, which has an odd number of 5's in the numerator (and an odd number of 2's in the denominator). So the square of a rational p/q can never equal 5/8.

In the second questionnaire, all students were given a page of basic facts about factorization (including uniqueness of factorization into primes). They all responded that they were familiar with these facts. They were presented with proofs C* and D* (half in the order D*,C*) and then asked if they understood either proof or were confused by either proof on the first read through. They were allowed to keep the questionnaires for a few days to see if their attitudes changed and, if so, were asked to record the change and when it happened.

Overall, 33 students responded to the frist questionnaire, 37 to the second out of a university mathematics intake where two thirds had at least one grade A in advanced level mathematics. There was no significant difference in the understanding (or confusion) between D and D* in the two samples, but C* proved to be highly significantly more difficult than C.

Amongst the students in the second questionnaire, D* was significantly preferred to C*, both in terms of understanding and (lack of) confusion. Most interesting of all, of those who claimed to have seen neither type of proof before, the preference for D* over C* was very highly significant. (Explicit χ^2 computations are given later.)

The passage of time improved the understanding of both proofs, but the preference for D* over C* increased still further. At the end of the response only three students out of 37 failed to understand D*. All of these had claimed prior acquaintance with proof C, stated immediately that they preferred C* and did not record any change of attitude. Meanwhile four other students who had seen C before and understood C* initially but not D* ended up by changing their preference to D*. There were no preferential changes from D* to C*. No student out of sixteen who thought about the matter overnight ended by stating a preference for C*; eight said that they preferred (or only understood) D*.

A number of reasons may be put forward to try to explain these results. For instance proof D* illustrates the reasoning with an example whereas C* merely repeats the ideas of C* whilst dropping the even/odd terminology. Proof D* deals with the roles of both prime factors 2,5 whereas proof C* consentrates on the prime 5 and neglects 2 completely.

A sophisticated view of proof would deprecate the use of an example, for an individual case proves nothing in general; it would also applaud the concentration on 5 because this is sufficient for the argument. However the learner sees things differently.

Students were asked to give the reasons for their responses. Time and again there was a reference to unfamiliarity causing lack of understanding and confusion, whilst familiarity was linked with understanding - a far cry from the strict application of logic being the criterion for a good proof. One would be justified in conjecturing that it was familiarity with proof C that was in large part responsible for the students' claim to understand it, suggesting an instrumental understanding rather than a relational one in some cases.

There are, however, sound reasons why proof D* is preferred to C*, despite the mathematical community's implied preference for C over D. The main reason is that proof D readily generalizes to demonstrate the irrationality of all square roots of non-squares, whereas proof C, though generalizable in spirit, is more specialist in nature because of the linguistic considerations. Thus proof D is *generic* in the sense that it contains within it a complete spectrum of proofs for all square roots of non-squares. It shows *why* no square of a rational equals 2 and the same proof readily adapts to 5/8 or any other non-square. Even the example $12^2 = 2^4 \times 3^2$ in D* is also generic in that it readily adapts to any other square. The proof C, on the other hand, achieves its ends by contradicting a subtle variant of the original statement. The idea of a generic proof ties in very naturally with the concept of "generic thought" described in [4].

General and Generalizable

It has been argued by Mark Steiner [4] that a proof of type D is more satisfying from a philosophical point of view than C. (I am indebted to Shlomo Vinner for discussion on this point.) Steiner argues that is is not the more general proof that is preferable, but the generalizable proof. He argues against the proof C and for the proof D on these grounds. He also puts forward the view that the more general proof of the type to be found in Hardy's "Pure Mathematics" does not give a better explanation just because it is more general.

Briefly, Hardy's argument is as follows:

Proof H There is no rational number whose square is m/n, where m/n is a positive fraction in lowest terms, unless m and n are perfect squares. For suppose, if possible, that

$$p^2/q^2 = m/n,$$

p having no factor in common with q and m no factor in common with n. Then $np^2 = mq^2$. Every factor of q^2 must divide np^2 and, as p and q have no common factor, every factor of q^2 must divide n. Hence $n = rq^2$, where r is an integer. But this involves $m = rp^2$; and as m and n have no common factor, r must be unity. Thus $m = p^2$, $n = q^2$, as was to be proved. In particular, it follows by taking m = 2, n = 1 that 2 cannot be the square of a rational number.

Proof H (all but the last sentence) comes from the textbook which revolutionized the teaching of mathematics in Britain earlier this century. The proof was included on the first questionnaire, along with proofs C, D. It is interesting to note that it produced less understanding and more confusion than either C or D to a level that in the sample concerned was highly significant statistically.

It seems that Steiner's thesis on the explanatory value of a proof from a philosophical viewpoint is supported also from a cognitive one. A generalizable proof (such as D or D*) works at the example level (such as 2 or 5/8) in such a way that the examples chosen are typical of the whole class of examples (and hence are generic). A general proof (such as H) works at a more general level (with the rational m/n) and therefore requires a higher level of abstraction.

Long Term Sophistication

So far we have considered the suitability of various proofs for beginners. This is not to suggest in any way that a proof suitable at this stage remains the most appropriate at all stages. When a higher level of abstraction is required, a more general proof may very well come into its own. For instance the technique of considering a fraction in its lowest terms (as in C or H) is a valuable one for more advanced work.

As an example, suppose that $f(x) = x^n + a_1 x^{n-1} + \ldots + a_0$ is a monic polynomial with integer coefficients. Then any rational root of f is an integer. A suitable proof is to suppose that the root is p/q in its lowest terms, then

$$(p/q)^n + a_1 (p/q)^{n-1} + \ldots + a_n = 0,$$

so

$$-p^n = a_1 p^{n-1} q + \ldots a_n q^n.$$

The right-hand side is divisible by q, so q divides p^n. Thus q is a factor of p and q and, since they have no non-trivial common factor, $q = \pm 1$ and $p/q = \pm p$ is an integer.

In this way the second principle mentioned earlier must come into play and the learner must be helped in achieving long-term sophistication. If this is done by using proof C as an introduction to proofs by contradiction, some degree of confusion is to be expected. This suggests that simpler contradiction proofs should be met first. Examples could include:

m^2 even implies m is even (for an integer m)
$0 < a < b$ implies $0 < \sqrt{a} < \sqrt{b}$ (for real numbers a,b).

Such effort would be worthwhile, for the contradiction proof C is easier to write in symbols than proof D, and with greater sophistication it can assume a more central role.

Responses to questionnaires

The 33 students responding to the first questionnaire understood the proofs, or were confused in the following numbers:

Total 33	D	C	H
understand	19	24	9
confused	10	6	23

Table 1

Comparing the understanding of D (19 understood, 14 no comment) with that of C (24 understood, 9 no comment) and calculating χ^2 (with Yates correction) gives $\chi^2 = 1{\cdot}1$, which is very insignificant (30% level). A similar conclusion holds for the level of confusion. However, as we shall see in the second questionnaire, a large number of students were likely to have been familiar with C, but hardly any with D, so in this test the rating of C is enhanced. Though C and D are not significantly different, comparing H with either gives a different tale. For instance, comparing H with C for understanding in the same way gives $\chi^2 = 11{\cdot}88$, which is highly significant (0·06%). Similar figures occur in comparing H with D.

The 37 students responding to the second questionnaire replied as follows:

Total 37	D*	C*
understand	25	14
confused	11	21

Table 2

More interesting were the responses of those who claimed to have seen neither type of proof before:

Total 20	D*	C*
understand	15	3
confused	4	17

Table 3

Comparing the total tables 1 and 2, we find that the level of understanding of D and D* are not significantly different (56%) nor do they cause significantly different confusion (96%). On the other hand, C* is significantly more difficult to understand than C (1%) and causes more confusion (0·3%). We can see that this is significantly affected by the large number of students (17) who had seen C before and, presumably were able to understand it and had no confusion from the start. In fact, subtracting the entries in table 3 from those in table 2 to obtain the figures for those who have seen C before (but not D), we find 11 of the 14 who claimed to understand C* have seen C before, whilst only 10 of the 25 who understand D* have seen C before. In this way we find that the effect of seeing C before has a highly significant effect on understanding C* (0·6%), whilst, as one might expect, it has no significant effect on the understanding of D* (30%). The figures for confusion are similar, having seen C causes a highly significant reduction in confusion for C* (0·06%) but has no effect on confusion in D* (29%).

The most important figures are found in table 3. For those students who have seen neither type of proof before, the level of understanding D* over C* is very highly significant (0·05%) and reduction in confusion for D* over C* is also very highly significant (0·02%).

Conclusions

We have seen in a group of 20 able students the significant preference in understanding of the proof D* over the proof C* when they have had no experience of either type of proof. The standard contradiction proof C, though mathematically elegant, lacks explanatory power and generalizes with difficulty because of linguistic considerations. The Hardy proof does not have greater explanatory power for students, despite its greater generality. Meanwhile the generic family of proofs including D and D*, though more verbal and less easy to write in symbols, has the elusive quality of explanation which enhances understanding.

The evidence suggests therefore that we should seek the explanatory power of generic proofs for beginners, rather than the aesthetic elegance or generality of general proofs. The latter can (and should) come at a stage when they are more likely to be appreciated.

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QUALITATIVE THOUGHT PROCESSES IN CLINICAL INTERVIEWS

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The clinical interview is a method of investigation where the investigator varies his questions according to the responses of the person being interviewed. Such techniques have been used with great success by Piaget and his school and are now used in a wider context which include various investigations into the nature of mathematical thinking.

Two major areas of contention arise in the clinical interview: the first is the interpretation of the interviews in a manner which is not purely anecdotal; the second is the nature of the interjections by the interviewer and their effect on the course of the interview. What is required is not just a study of the specific factors in each interview but a general set of principles which apply to a broad class of interviews.

Such principles could be based on a mathematical analysis of the ideas involved, but this tends to superimpose a logical framework which is at variance with the observed manner in which thinking actually proceeds. An alternative is to consider the qualitative nature of the thinking processes: the continuous stretches of thought, the mental leaps, conflicts, blockages, and so on.

Here we consider such an analysis which proved amenable to describing a number of clinical interviews which arose when I was privileged to join in with a working group on problem solving at Concordia University, Montreal. It is based on an underlying resonance model of brain activity and is proposed here as a basis for discussion. Should it prove fruitful, reinfinements would be welcome,both in terms of the nature of the analysis itself and the names used for the various phenomena (which are still in need of reorganization and clarification).

First we begin with some observed phenomena:

1. Initial resonances: immediate responses to stimuli which occur without time for reflection. If consonant, such a resonance may provoke an ongoing schema but if dissonant elements are present to a sufficient degree there may be a conflict leading to an alternative schema or a mental blockage.

2. Ongoing schemas: Connected chains of thought without major crises or changes in direction. In clinical interviews the extent of such schemas are often difficult to determine because of the interjections of the interviewer. We may wish to distinguish certain types of schema e.g.

a resonance schema: carried on by the intense power of the thought process itself. Such a schema is not disturbed significantly by external comments. It may end up with the subject losing track of the original problem because the goal of solving the problem has been temporarily lost by the the internal compulsion of following through the schema itself.

superposed schema: where the initial resonance involves two disparate schemas which produce a novel train of thought superimposing two (or more) resonances. The subject may feel that such a superposition will lead to a solution & does not (yet) sense a definite conflict.

conflict schema: the subject realises that a conflict is present, but has not, as yet, resolved it. He may flash from one idea to another.

explanatory schema: an explanation of thought processes which have already taken place. At this time questions from the interviewer will probably be more welcome than in, say, a resonance schema.

other types of schema? e.g. an open search schema?

Schemas are, broadly speaking, continuities in thought (though they may involve small discontinuities not noticed by the observer).

3. Qualitatively different are observable discontinuities:

conflict causing disruption of thought
mental blocks: stoppages in thought, often preceded by conflict.
insights: sudden leaps in thought.
finish: end of schematic action.

These discontinuities (particularly the first three) are of extreme interest in clinical interviews. We need as much information as possible to interpret them (tape, audiovisual aids etc.)

The flow of thought is often affected by an interjection from the interviewer, indeed, this is part of the special function of a clinical interview. Interjections occur in two different ways:
(i) as a response to a question from the person being interviewed,
(ii) as an unsolicited interjection.

A comment of type (i) is a natural part of the flow of thought, for the person being interviewed is seeking a reaction from the interviewer. The response may be of many kinds, it may be clinically neutral, it may support the student's chosen path of attack, or disagree and turn it along different lines. At the other extreme it may be a 'torpedo' (a slang term meaning a provocative and startling challenge). However this is often unsolicited (!) and should probably always be reclassified under type (ii).

The comments of the second type are fundamentally different; they may occur at a time when the student does not wish to be disturbed and they may even disrupt the thinking process. For instance a question in the middle of a resonance schema may upset the train of thought and a response to such a question may not evoke a true explanation of the thinking process, the latter may not be a verbal process anyway. Interjections on the part of the interviewer can have strong effects at critical points in the thinking process. Even a murmur of approval or a non-verbal piece of communication can affect the outcome. (Hence the inadequacy of written protocols in clinical interviews.)

It is essential to take ideas such as these into account in deciding general modes of behaviour on the part of the interviewer on a broad class of interviews. (These modes may change, provided that their nature is made explicit; for instance, in one class of interviews it may be decided not to interrupt resonance schemas, but, in another, resonance schemas might be purposely interrupted to study the effect of such an action. Or a tape of the interview might be played back to the subject, seeking information about what might have been happening internally at a certain time, though one suspects that this will only lead to a post-rationalization of thought rather than a description of the thought itself.)

Some General Thoughts for Discussion

1. Initial resonances are often overlooked as meaningless remarks before thinking is settled down, on the contrary, perhaps they give essential clues as to the nature of the thinking process, e.g. they may involve over-extrapolation from the evidence leading to later conflict.

2. From observation ideas expressed in one resonance schema may conflict with those expressed in a subsequent resonance schema but this conflict is not observed as such by the subject provided the ideas are locally coherent. Perhaps conflicts are local, not global, mental phenomena.

3. Local errors (stumblings, immediately corrected) may not have significant effects on the overall flow, in general, (a case of "structural stability") yet could have extreme effects at critical points in brain activity.

4. Diagrams drawn by the subject are more permanent than verbal comments (which may be forgotten), in particular, verbal comments made in one schema may not be remembered during the course of a later one.

5. Does the memory of something earlier depend on the level of energy involved in the earlier mental activity in some way? (mental activity involves electrical brain activity. Potential differences in different parts of the brain which lead to the passage of electrical current yield a potential function. Structurally stable brain activity may involve local minima of the potential function. The changes of equilibria in time may yield continuous schema and discontinuous leaps. Memory may depend in some way on

the level of energy in the earlier activity.)

6. Conflict in problem solving protocols may be different from conflict in Piagetian tests. In problems the conflicts may be caused on occasion by transient electrical phenomena which when questioned cannot be recalled. In Piagetian transition there are usually two fairly stable (chemical) mental configurations causing the conflict; the subject is able to talk about each one separately in a suitable context.

7. Words like "carelessness", "mistake" etc. are a vocabulary from a logically based paradigm of interpretation. A 'mental resonance' concept may be more suitable for description of thinking processes.

8 It is known from electroencephalograph readings that brain activity becomes progressively more complex with maturation. Some young children do not always seem to notice conflicting results in their mathematics. With more complex brain configurations mature students can experience very subtle conflicts. Are there genuine qualitative differences in brain activity at different ages?

9. Mature mathematicians are able to carry through subtle proofs and understand each step yet feel uneasy about the final result. Does this unease signal subconscious cognitive conflict? Are "subjective emotions" sometimes a better guide to the coherence, or otherwise, of brain activity than "observable ability to carry out mathematical processes"? The brain is a complex mechanism and in the presence of competing resonances, it may well happen that mathematical process may be carried out coherently in one resonance system and it is only the emotional unease which signals the underlying conflict. Whatever the pitfalls surely clinical interviews have much to offer that cannot be found by psychometric methods.

THE CONCEPT OF COMPLEX NUMBERS: AN EXAMPLE OF ACCOMMODATION IN THE LEARNING OF MATHEMATICS

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1. Learning Theories and Mathematics Learning

Mathematics learning is one of the most complicated types of learning. There are very few learning theories which relate to complicated types of learning and therefore can be applied to mathematics learning. Among these there are the theories of Piaget and Ausubel. An additional learning theory, directly oriented toward mathematics, is suggested by Skemp in (4) and (5). Some of the notions used in the above theories have quite similar meanings and few of them are even common (although not having exactly the same meaning). Such notions are assimilation and accommodation. Notions that have similar meanings are scheme (in Piaget), idea and subsumer (in Ausubel) and schema (in Skemp). Each of the above theories has some illustrations which are supposed to give meaning to these notions but, in general, when we are in a specific learning process (especially in mathematics) it is very hard to tell what the schemes of the learner are and why or how accommodation and assimilation take place(1). It seems that in order to make learning theories more meaningful and more applicable (especially to mathematics learning) it is necessary to relate to specific learning processes and to analyze and explain them in terms of the theories. In other words, the psychological theories should be systematically used in order to analyze and explain phenomena in mathematics learning (whereas, in most cases, this has been done with simple examples where the purpose was to give interpretation to the terms of the theories or to illustrate them).

In this paper we deal with the concept of complex numbers from the above point of view in relation with the notion of accommodation.

2. Two Types of Accommodation

In (1), p.117, Ausubel claims that "the learner is simply required to comprehend the material and to incorporate it into his cognitive structure." However, very often and especially in mathematics the cognitive structure of the learner is not suitable for incorporating the new material. In this case the cognitive structure has to undergo some changes, which is accommodation.

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(1) In Skemp (4), pp. 225-226 there is a brief description of assimilation and accommodation involved in the learning of the concept of number. But one can conclude immediately that this is only a short illustration because such a complicated concept, the learning of which takes years, cannot be described in two pages.

There are two types of change which we would like to discuss here. The first type is a relatively small change in the existing scheme. The second type is a complete substitution of the existing scheme by a new scheme which is totally different from the old one.

As an example of the first type, one can think of the following: A child learned the concept of positive integers in such a way that in his cognitive structure a positive integer was identified with the number of objects in a certain concrete set (chairs, beads, apples, etc.). Later on, this child was supposed to learn the concept of zero, namely, he had to relate it to the above cognitive structure (positive integers as cardinal numbers). In order to do that, he has to change his existing scheme for positive integers and to extend it also to situations where empty sets are considered, or, in non-technical words, to situations where one has nothing of certain species (no chairs, no beads, no apples, etc.). Although this is a relatively small change, it is not necessarily easy for the child. Perhaps an expression of the difficulties some people have with zero is the well-known declaration that zero is not a number.

An example of the second type of change can be the same child when he has to learn the concept of negative integers and his teacher uses the number line as a means of instruction. In this case the child has to substitute his existing scheme for non-negative numbers (cardinal numbers) by a totally new scheme, namely, an integer represents a point on the number line.

Following the terminology in (1), pp. 125-134, we will call a change of the first type by subordinate accommodation and a change of the second type by a superordinate accommodation. (2)

3. The Case of the Complex Numbers

The topic of complex numbers is included in some high school programs (high level mathematics courses) and also in some mathematics courses for science students. Of course, it is a part of every mathematics major student curriculum.

Sometimes, there are two alternative ways of teaching a new concept; the first requires a subordinate accommodation whereas the second way requires a superordinate accommodation. The use of the complex numbers is like that. We can teach them by saying that there are numbers with which we have not been acquainted till now, one of them is denoted by i and its square is equal to -1. (3) Other numbers are obtained by multiplying i by the well-known numbers (the real numbers). The result has the form bi (b is a real number). If we add the last result to a real number the result will have the form a + bi. The numbers that have the last form are called complex numbers. The complex numbers yield all the laws concerning addition and multiplication of real numbers and therefore: $(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$ and also $(a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1)i$. This approach reflects the idea of extending the field of real numbers by an adjunction of a root of the polynomial $x^2 + 1$. (4) The way it is presented here is probably not acceptable from the mathematical rigor point of view (as a matter of fact, the above equalities are definitions and not laws). However, it is quite intelligible for students who are not mathematics majors. Moreover, it requires relatively small change in the student schemes since he can use, for instance, $5 + 3\sqrt{2}$ as an analogy for $5 + 3i$ and since the main fact to which he has to get used is that there is a number (of which he has not been told till now) the square of which is -1. Hence, this way of teaching requires a subordinate accommodation.

Another way of teaching the complex numbers is to form all pairs of real numbers. Two pairs will be considered as equal if they are identical. Addition and multiplication of pairs will be defined by the following equalities, respectively:

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(2) Skemp, in (5), calls them expansion and reconstruction respectively.

(3) This approach was recommended in (6).

(4) For details, see, for instance, Van der Waerden (7), Chapter V.

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 - b_1b_2, \quad a_1b_2 + a_2b_1)$$

Then one can prove that the above collection of pairs is a field extending the real field. Moreover, it has an element (the pair (o, 1)) the square of which is (-1, 0). (5)

It is clear that in order to assimilate the concept of the complex numbers as it is introduced in the second approach, one must change completely his schemes for numbers. He has to assimilate the idea that numbers are "man-made', that mathematicians 'arbitrarily" define mathematical objects and mathematical operations and that the real numbers can be identified with all pairs having the form (a, 0) and therefore there is a number the square of which is equal to -1 in the complex field. Hence, this way of teaching complex numbers requires a superordinate accommodation.

Although it seems that a teaching method which requires a subordinate accommodation will be simpler for the students than a method which requires a superordinate accommodation -- this is not necessarily the case. Since we know very little how accommodation occurs, we do not know whether in some cases it won't be easier for some students to replace a whole scheme by a new one than to change it a little. Therefore, the decision which way is the best for teaching the new concept is quite hard.

4. The Sample and the Questionnaire

Our first goal was to examine accommodation in the learning of complex numbers in three different groups. The first group consisted of 14 high school students in Jerusalem who took part in an enriched mathematics program. The high school itself is very selective and therefore this group is definitely above the average of regular math. students in the whole country. The second group consisted of 39 science freshmen who had to take a short course in Linear Algebra as a preparation for further studies in Chemistry or Biology. The requirements in this course are not high and the level of the course is officially announced as B-level. The third group consisted of 13 college students who were essentially similar to the students in the second group. The only difference was that the level of the course they took was officially announced as A. As a result of this, the group was more mathematically oriented than the second group and the students were much better. Each group had a different teacher who taught the subject of the complex numbers according to his own mathematical and pedagogical approaches. The teachers of Groups 1 and 2 taught it in the second way (complex numbers as pairs) whereas the teacher of Group 3 taught it in the first way (adjunction of a number the square of which is equal to -1).

Using comments of students who came to us to discuss the difficulties they had with the subject, we compiled a questionnaire which was distributed to the three groups. This occurred at least 5 weeks after the subject had been taught.

The questionnaire was the following: "After complex numbers were taught in a certain class, the teacher asked his students to express their feelings and opinions about the subject in the form of a free composition. Here are two answers of the students to which you are asked to react in the following way: At the end of each answer a line segment will be given divided into 5 subsegments denoted by the numbers 1 to 5. Number 1 for: "I absolutely disagree" and number 5 for: "I absolutely agree." The other numbers are for other degrees of agreement or disagreement according to the following key:

2 - "I do not agree but I do not completely reject it."

3 - "I agree partially but also disagree partially."

4 - "I almost agree."

Please choose one of the subsegments according to your degree of agreement or disagreement and make a short comment about your choice.

A. We were told that i denoted a number the square of which was -1 but we were not told what that number was. In this respect, we lack an essential information.

B. Till now every number was written by means of digits. All of a sudden the number i is introduced to us and nobody tells us what its digits are. Certainly there is a way to express i by means of digits but this way is probably very complicated and therefore was not introduced to us."

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(5) For details, see, for instance, Caratheodory (2), Sections 2-9.

The students were told that the questionnaire was not a test and there are no correct or incorrect answers to it. They were asked to write what they really felt about the subject. They were asked (in addition to what was written in the questionnaire) to make short comments and not to write their names on the questionnaire if they did not feel like doing it. They were told that their answers can be used to draw some conclusions about their mathematical concepts and if they did not want to treat the questionnaire seriously they had the option not to fill it at all. Our impression was that the questionnaire was seriously treated.

The answers were analyzed and categorized both according to the answer profile (namely A2B3, A1B1, etc.), and to the written comments. We avoided decisions about questionnaires that lacked written comments.

5. Different Stages and Types of Accommodation

In the accommodation process we decided to distinguish between four different stages:

Stage I: No accommodation
Stage II: Partial accommodation
Stage III: Almost accommodation with slight expressions of awkwardness.
Stage IV: Complete accommodation.

We would like to say immediately that this distinction is more theoretical than practical because quite often it is hard to determine the exact stage of a specific student. Moreover, the distinction is more subjective than objective because it reflects our point of view about how accommodation should be. It is a mathematician's point of view. A student considered by us to be at the stage of partial accommodation might feel that he fully understands the subject, but mathematicians and teachers might discover inconsistencies and expressions of awkwardness in what the student says. It is us who think that his understanding is not satisfactory. Hence, the criterion we suggest for classification is subjective.

In addition to the above stages we also categorized the answers according to the types of accommodation. These categories describe segments of the accommodation process. In each stage several categories might appear and they can last more than one stage in the process. Certain categories occur mainly in the beginning of the process whereas other categories occur mainly at its end. The relation between the different stages of accommodation and the different types (categories) of accommodation will be shown in Diagram I.

As to the categories above, we would like to say that here two kinds of work were done. The first one was to identify the groups of answers that had something in common. The second one was to suggest psychological characterizations for these groups. Hence, the first kind of work is more objective whereas the second one is again more subjective. But it seems that there is no way of avoiding subjectivity in the beginning of this type of research. As Easely said in (3), p. 289 (about very similar analysis concerning cognitive structure): "It begins in subjective but hopefully educated judgments and moves toward objectivity."

To those readers who do not like the subjective approach (or like only their subjective approach) we suggest the more objective part of this paper, namely, the group of similar answers. They are open to those readers' interpretations.

Among all the above categories, there was also the category of students whom we could not categorize because of lack of written comments.

In the examples we are bringing here, we will first state to which group the student belongs, then his answer profile (A2B3, for instance, means that his choice in A was 2 and his choice in B was 3), then parts or the whole of the written comment will be quoted, and then, in parenthesis, we will state the student's stage of accommodation (according to our judgment). Sometimes we will add to this our comments, but not always. If no other considerations are given, the key for determining the stage and the category is the following: An answer profile AjBK with j > 1 or K > 1 is considered as a (full or partial) support in the views expressed in the questionnaire. The written comments should be used than to determine whether this support is a result of conceptual difficulties or it is a dissatisfaction with the methodology presumably underlying A and B in the questionnaire.

One answer can belong to more than one category. The reason for its appearing in one category and not in another is that we wanted to emphasize only one aspect of it. As we said before sometimes we bring only parts of the student's answer but the decision about his stage is made on the basis of the whole answer, so sometimes the reader will lack certain information concerning the decisions about the stage.

Category (a): Explicit announcement of difficulties

Group 1, A4B4: Many things are missing and time is short. Perhaps certain materials which are needed for the understanding of i are missing (Stage 1).

Group 2, A2B2: They just threw this number at us although we were used only to ordinary numbers... We could not understnad how a square of a number is equal to -1. (Stage 1).

Category (b): Avoidance of personal responsibility (here justifications are achieved by referring to the authorities (teachers, books). The student will say: "We were told that..." or "They explained to us that...").(6)

Group 2, A4B2: They explained to us that it is not a real number, as if it had an axis of its own, not the x-axis or the real number axis... (Stage I-II).

Group 2, A5B1: "We got clear hints how to use it in computations (Stage I-II). (Note the (potential) conflict between A5 and the "cooling down" . statement that we know "how to use it in computations").

Category (c): General and almost non-relevant statements

(The student comes up with something like a declaration that hides his difficulties. Instead of a direct confrontation with the problem comes a general statement and the question is suppressed.)

Group 3, A4B1: This number enables us to extend the domain of problems (we can deal with) by problems where we have negative numbers under the root sign. This was either impossible till now or very complicated.

Group 2, A1B1: By saying that i denotes a number the square of which is -1, it makes no difference what this number is. Moreover, such a number does not have to exist because mathematics is a game the rules of which are invented by the scientist... A number is a definition that can be interpretated in many ways and everyone chooses the interpretation that fits him best (Stage II).

Category (d): This is how it is

(here the student understands the requirements and he is disciplined. He plays the game as it should be played, but he does not show any personal concern.)

Group 1, A1B1: It is impossible to define i by means of ordinary concepts; i can be exemplified either theoretically or by means of exercises for which it is a solution (for instance, quadratic equations that have a negative discriminant). (Stage III).

Group 1, A1B1: It is true that we still do not use i freely but we do not lack information. It is impossible to express i by means of digits since the ordinary digits can represent a location on the number line but i cannot (Stage III).

Group 2, A1B1: It does not make any difference how a thing is represented. As far as I am concerned I could have represented a certain number by a rectangle (Stage III).

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(6) Our impression as teachers is that very often when a student says: "The teacher told us", "We were shown that...", etc., he cannot intellectually cope with the difficulties. He, therefore, prefers not to deal with them and quotes the authorities instead. The above expression should be a red light for teachers.

Categories (b), (c), (d) are closely related to each other. Each one of them (except (b)) can be considered as a development of the previous one in the sequence. Hence, the distinction between them is not always so clear.

Category (e): Rejecting i as a member of the "Number World" and emphasizing its computational aspect (It exists only in computations).

Group 2, A1B1: i is simply a technical means to overcome certain deficiencies in mathematical structures. These deficiencies are revealed in several stages but during the computation process either the difficulties are resolved or a real result is obtained or i^2, i^4 appear (Stage II). (Note that i is only a tool to overcome difficulties. The moment the difficulties disappear i also disappears. Therefore, there is no need for further information about i and the answer profile is A1B1.)

Group 2, A4B2: It is not exactly a number. It is a tool by means of which one can make certain manipulations in various mathematical domains (Stage I-II). (Although manipulation is emphasized, there is still the need for more information expressed by the choice A4).

Group 2, A2B2: We were not told what this number was but it is absolutely possible to manipulate it even without knowing exactly what it is. In the beginning one really thinks that i can be written by means of digits but later on, it turns out that it is possible to express it not exactly by digits but in other simple forms (Stage III).

Group 1, A1B1: i is not a number and no information is missing. Imagination is missing (Stage III).

Category (f): Private analogies

(Here a very personal effort is made to accommodate by means of analogy. The private analogy is meaningful to its discoverer although it might be mathematically mistaken.) (7)

Group 1, A4B2: In my opinion this expression is, probably, "not logical". Therefore, it is impossible at all to express it in numbers, like 1/0 (Stage I).

Group 2, A5B2: Some numbers are known like the infinite non-periodical decimals about which it is not known how they behave after a while and there is no need to express them in the common way since numbers like them are not known. (Stage I). (It is interesting that the student does not consider the form of infinite non-periodical decimal as a digital form. The statement is quite confused. It is not clear whether numbers of a certain kind are known or not known. In spite of the analogy, the choice is A5, which, probably, shows that the analogy is not quite convincing.)

Group 3, A2B1: i is the number $\sqrt{-1}$. It is more convenient to denote it by a letter, as it was with π (Stage III-IV).

Category (g): i is identified with $\sqrt{-1}$. $\sqrt{-1}$ is considered as a digital expression for i. i is identified with a vector on the imaginary axis, namely, it has a geometrical interpretation. (All of these can be considered as a subordinate accommodation. The new concept is reduced to some familiar concepts.)

Group 1, A1B1: We have all the information about the number because we have an equality sign by means of which we express i using known terms such as $\sqrt{\ }$ and -1 (Stage IV).

Group 1, A1B1: The basic information about i is that it is equal to $\sqrt{-1}$. i itself is a digit, only in a different domain, and it can be used as it is (Stage IV). (In addition to the aspect of Category (g), this answer has also an aspect of Category (e). The remark that i itself is a digit indicates an interesting direction of accommodation, namely, by extending the concept of a digit)

Group 1, A1B1: ... i was introduced by means of real coordinate system I R that provided a concrete representation for those who needed it (Stage III).

These were the main categories we found in this sample. Their relation to the 4 stages of accommodation is shown in the following diagram:

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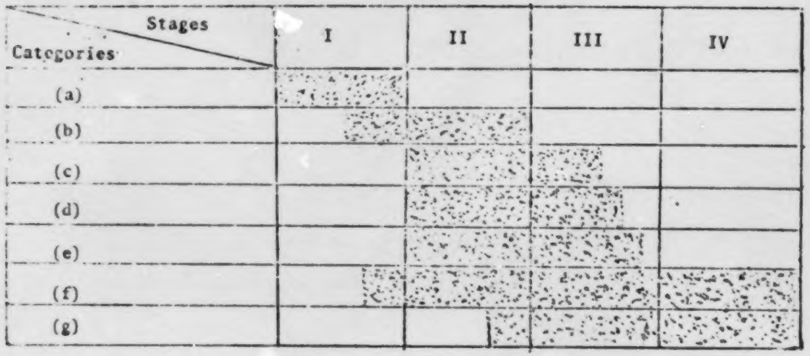
(7) In fact, some new chapters in science and mathematics started as private analogies.



In the above categories three major types of accommodations can be distinguished:

(1) External accommodation (Categories (b), (c), (d))
(2) Operational accommodation (Category (e))
(3) Subordinate accommodation (Category (g))

Superordinate accommodation hardly occurred in our sample. Only one student (out of the two groups to which complex numbers were introduced by means of pairs) mentioned that complex numbers are pairs of real numbers and i is the pair (0,1).

The most common way of the subordinate accommodation was to identify (somehow) i with $\sqrt{-1}$. It is worthwhile to mention that the expression $\sqrt{-1}$ hardly occurred during the teaching in the above sample. (It appeared only in the solutions of quadratic equations that had negative discriminant.) In spite of that, this was the most common way to consider i. It only shows us that the student himself very often chooses how to accommodate, not necessarily the way the teacher wants it.

When considering the numbers and percentages of students in the different stages of acccamodation we considered Stages I, II together, denoting them by NA (No Accommodation), and Stages III, IV together, denoting them by A (Accommodation). By doing this we decreased the number of cases in which the decision how to classify them was hard. Even so, there were few cases in which it was hard to decide whether they belong to Stage II or Stage III. Hence the table will give only a rough estimate. The group of answers that lacked written comments will be denoted by NC (No Cooperation).

Table I

	NA	A	NC
Group 1, mathematically oriented high school students, complex numbers taught by means of pairs. N = 13.	6 (43%)	7 (50%)	1 (7%)
Group 2, Science students, level B, complex numbers taught by means of pairs. N = 39.	15 (38%)	9 (23%)	15 (38%)
Group 3, Science students, level A, i introduced as a number the square of which is equal to -1. N=14.	4 (31%)	9 (69%)	0 (0%)

The same questionnaire was distributed also to another sample that consisted of math. major students. The additional categories of accommodation found there were all superordinate. The percentage of students who achieved full accommodation was much higher there than in the first sample. Details about the second sample will be given elsewhere.

8. Discussion

Table I shows that the percentage of those who achieved accommodation in group 3 is greater than the percentage of those who achieved it in group 2. This difference is also statistically significant at the $\rho .05$ level (when using χ^2). On the other hand no significant difference was found between group 1 and group 3. However, accommodation in all three groups was mainly subordinate. This suggests that the teaching method which requires subordinate accommodation is preferable in our case. Of course, one can argue that the difference between group 2 and the other two groups might be a result of the relative weakness of that group and it is not a result of an unsuitable teaching method. To support this view one can also point at the fact that no significant difference was found between groups 1 and 3. Our answer to that is the following: Groups 1 and 3 are groups of good students and for good students teaching (as long as it is reasonable) does not make a difference. A good student can find his way to accommodate even if concepts are not introduced to him in a way that fits his cognitive structure. As a matter of fact, this is what happened with group 1. To this group complex numbers were introduced by method that required a superordinate accommodation but the students accommodated in a subordinate way. Something like that can hardly occur with average or weak students. If concepts are introduced to them in a way that does not fit their cognitive structure they get lost. For average and weak students teaching methods do make a difference. Anyhow, since superordinate accommodation hardly occurred in our sample one can seriously question whether the pairs method should be used when teaching students like those we had in our sample (namely, non-major math students).

Moreover, if we generalize what we have said to other learning situations we would recommend the following:

If there are two alternative approaches for teaching a new subject, the first one requires a subordinate accommodation and the second one a superordinate accommodation, and if in future learning of the class the superordinate approach won't be needed -- then it is better to use the subordinate approach.

If there is only a superordinate approach for teaching the subject and the superordinate ideas won't be needed in future learning of our students then, perhaps, one should reconsider whether such a subject should be included in the curriculum at all.

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Mathematical Variables and Verbal "Variables": An Essential Difference

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One fairly common way of helping students understand the use of literal symbols as mathematical variables (n, x, a, ...) is to draw an analogy between mathematical variables that act as generalized numbers (Küchemann, 1978) and ordinary words that act as generalized persons or objects in everyday language. While it is undoubtedly helpful to students to point out the similarity between mathematical variables and ordinary language "variables," it is also important to draw attention to the differences between these two types of variables so that students can better understand the concept of variable and, at the same time, appreciate the power which mathematical variables have beyond that of ordinary language "variables."

The similarity between mathematical variables and verbal "variables" lies in the use of a variable as a symbol, or placeholder, to represent individually, yet simultaneously, each element of a replacement set in some statement or assertion about the elements of that set. For example, in the statement, "If n is an integer, then $2n$ is an even integer," the variable n is used to represent individually, yet simultaneously, each element in the replacement set of integers. For this particular statement the replacement set happens to coincide with the truth set, the set of elements in the replacement set for which the statement is true. In general, the truth set is a subset, often a proper subset, of the replacement set, as in the case of the statement, "If n is an integer, then $2/n$ is a rational number."

Many words and phrases in ordinary language function in certain contexts in a way similar to literal mathematical variables. For example, pronouns, some proper names, many noun phrases, and certain special words can be used to represent elements of a replacement set, as illustrated in the following statements:

He teaches mathematics.
John Doe thinks so.
An even integer is divisible by 2.
This box contains *something*.

In these statements the words and phrases *he*, *John Doe*, *an even integer*, and *something* all function as variables in the sense that the names of various elements can be used to replace these words or phrases to obtain true (or false) statements.

It has been claimed (Hockett, 1967; Russell, 1940) that mathematical variables behave exactly the same way as ordinary language "variables." It has also been said (Beberman & Meserve, 1955) that the usefulness of literal variables is based on their ability to represent unspecified members of a set, a role that ordinary words and phrases can perform, albeit awkwardly at times (Tarski, 1941).

In fact, there is an important difference between mathematical variables and verbal "variables." The literal variables used in mathematics are not encumbered with the connotations attached to the words and phrases used in ordinary language. The real power of mathematical variables resides in this semantic neutrality, a feature that, in turn, implies a completely independent relationship between mathematical variables and their associated replacement sets. It is the independent relationship between literal variables and their replacement sets that gives mathematical variables a generality and flexibility that verbal "variables" do not possess.

The generality of mathematical variables stems from the fact that the use of a particular literal variable, such as x, in no way restricts the replacement set associated with it. Consequently, changes in the replacement set do not necessitate changes in the literal variable. In the case of verbal "variables," the choice of any particular word or phrase can place an inherent restriction on the replacement set, thereby limiting the generality of the statement. For example, in the sentence "*He* teaches mathematics," the replacement set for *he* is restricted to human beings of the male sex. If the replacement set needs to be expanded so that females can be considered, the "variable" he must be changed to a form that can represent both males and females. Mathematical variables, by way of contrast, can be selected without regard to the replacement set, and a single statement can be used, once and for all, to assert a fact or relationship. The sentence, "X teaches mathematics," can refer not only to both males and females, but even to computers. A mathematical variable, like x, carries no connotation in and of itself and thus does not limit in any way the choice of replacement set. For this reason mathematical variables allow more generality than verbal "variables."

Conversely, the flexibility of mathematical variables stems from the fact that, given a particular replacement set, a potentially unlimited number of nonconnotative, interchangeable variables are available for use with that set. Consequently, changes in literal variables do not necessarily imply changes in the associated replacement (or truth) sets. For example, using real number values for x and y, the statement $y = 2x$ describes exactly the same functional relationship as the statement $z = 2w$. Changes in verbal "variables," on the other hand, do imply changes in the associated replacement sets because of the unavoidable nuances of connotation attached to different words and phrases. Because mathematical variables carry no connotations, they are arbitrarily interchangeable and thus allow a flexibility far beyond that of verbal "variables," a flexibility which is essential for operations like composition of functions (Wagner, 1977).

It has been shown (Wagner, 1977) that many students do not understand the arbitrary nature of mathematical literal variables. Particularly, many students tend to believe that different letters used in the same context represent different values. For example, even some students who have studied algebra do not seem to realize that changing the "unknown" in an equation does not affect the solution to the equation. It might be helpful, especially for these students, if teachers emphasized the differences, as well as the similarity between mathematical variables and verbal "variables." By pointing out explicitly that mathematical variables and replacement sets are mutually independent, that a variable does not restrict its replacement set and, conversely, that a replacement set does not limit the choice of variable, that changes in the replacement set do not necessitate changes in the variable and, conversely, that changes in the variable do not necessarily imply changes in the replacement (or truth) set, perhaps students will begin to gain a better understanding of the generality and flexibility, the real power of mathematical variables.

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LISTENING-IN TO PROBLEM SOLVING

F.R.Watson.

This paper discusses the issue of what we can deduce from transcription and other records of pupils' problem solving activities.

Bruner (1956) refers to the difficulty of studying thinking, and the need to externalise thinking behaviour - to "get it out into the open" where we can look at it. Thinking is a peculiarly private activity, and particularly in the complex tasks commonly described as problem-solving, an attempt to describe the process whilst simultaneously engaged *in* it is liable to cause interference. Aside from this, as Donaldson (1963) has remarked, pupils are *not* trained observers of cognitive behaviour, and most subjects are likely to find difficulty in describing the progress of their thoughts, as distinct from speaking aloud the thoughts themselves. Thus Donaldson encouraged her subjects to 'say' their answers to the problems, as did Krutetskii (1976). The work referred to below has adopted a similar approach. The "psychometric" approach, typically involving large samples, written responses, and often short questions, has many advantages - in particular those of representativeness and the avoidance of subjectivity. Yet it does seem inappropriate to the study of problem-solving *processes* - for we are concerned not so much with the results achieved by individuals, but with the detailed study of the route by which these results are arrived at. As Krutetskii writes

> "(Written) tests establish only the final result of an examinee's execution of a certain task, ignoring the nature of the actual process of attaining that result... (they) do not provide a complete picture of the phenomenon under scrutiny." (1976, p 13)

The alternative is to resign oneself to working with small samples, and to take the stance of trying to interpret the evidence of pupils' speech, gestures, facial expressions, etc. as well as what they write or draw.

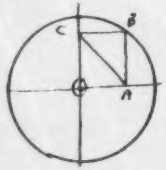
Such interpretation is frequently idiosyncratic and subjective, although some strategies have been suggested which can help to reduce the uncertainty surrounding such judgements. An obvious one is a 'second opinion' - so that tape-recording is particularly helpful here (and brings the incidental benefit of allowing us to analyse at leisure and more than once). Another is referred to by Charlesworth (1964) in connection with Piagetian studies in cognitive development - the introduction of an element of surprise. Children were tested for conservation using a conjuror's 'magic box' with a secret compartment. Those who had attained conservation were surprised when the number of coloured balls in the box was not what they expected; others who had not reached this stage saw nothing unusual in the situation. (At least, they *appeared* not to do so - only the *positive* evidence of surprise is valid here!) (A similar instance of surprise indicating the course of a pupil's thought may be found in Watson (1974)).

Work has been done at the University of Keele on studies of problem-solving and of children's thinking in general. Eagle (1973) gave an account of children's attempts at three-dimensional problems which was part of a larger, unpublished investigation. Thorpe (1978) used interview methods in a further study of this topic. Rose (1977) in an M.Sc. dissertation studied problem solving in mathematics at a variety of levels from first year Secondary school (age 11 years) to postgraduates and mathematics lecturers; his work contains several transcripts of the course of problem-solving from which inferences may be drawn - an example of such a transcript is given below.

So far we have not defined 'problem' - it will be taken here to mean a mathematical exercise of a non-routine character which, if the subject is successful, may be solved in a period of (say) 10-20 minutes. Many of the exercises used by Krutetskii were standard 'text-book' exercises - some of considerable complexity (not unusual in Russian schools), but mostly such as would yield to the application of previously-encountered normal classroom methods, and often embedded in a context of somewhat similar exercises. Thus their difficulty lies partly in the fundamental understanding of the priciple involved - eg. in recognising $a^2 + 2ab + b^2 - C^2$ as the 'difference of two squares' and partly in the degree of complexity of the exercise.

Some of the 'problems' used by psychologists in their investigations might be classified as 'tricks' - for example the pendulum problem of Maier (1926) and the well-known one illustrated in fig. 1 below.

0 is centre of circle
OABC is a rectangle
AC = 5 cm.
Find the radius of the circle.



These demand what we may describe as an intuitive leap in thinking but once the solution is seen it is immediately apparent and no further work needs to be done - an all-or-nothing situation.

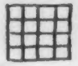
Problems as we define them demand some investigation, as in the case below; where the method of attack is not necessarily obvious and several steps must be taken before the 'pattern' of solution is apparent. Such investigations may be developed at great length - see eg. Banwell (1972), Bell (1977), for description of their extensive use in teaching.

Our 'problems', however, have a smaller time scale, and although some of them might be suitable starting points for investigations, the latter are usually more 'open' and may extend over a larger period of time - days or weeks.

The aim of the research undertaken has been in the first instance to provide a descriptive account of what pupils and students actually do when they solve problems. Clearly, what is noticed and inferred will not be independent of the predispositions, hypotheses and interests of the researcher, but if one accepts that, at present, we appear to know very little about how problem solving is accomplished, a first task seems to be an attempt to gather data in as open a way as possible, upon which to build hypotheses, rather than to formulate and test hypotheses by large-scale comparisons in advance of an adequate conceptualisation and description of the processes occurring.

In the example below an able third-former in a grammar school was asked to solve the problem -

How many squares are there in this diagram - of all sizes?
How many would there be on a chessboard (8 by 8)?



Fortunately, if untypically, this subject was quite willing to "chatter" into the tape recorder as he worked. He sat alone in the room once the problem had been handed to him - what he said as he worked is given in the transcript. Encircled numbers refer to the notes which attempt to elucidate or comment upon his remarks. The activities observed here may be compared with the problem-solving procedures suggested by Polya (1957), Schoenfeld (1975) and others.

A first concern is to see what behaviour we can identify and to what extent we can categorise the behaviours which emerge, their relative frequencies and inter-relations, and whether we are able to interpret the course of the solution. The extract below gives some grounds for optimism, - there is a rich collection of observable behaviour here - and experience suggests that few 'interviews', properly set-up, fail to provide at least some information.

The types of behaviour we may note are those of checking, conjecturing, planning, symbolising, giving a reason for a statement, making use of mathematical background, and so on. (Some of the notes are for clarification of what was said, or explanation of relevant external circumstances - they help us to make sense of the record, though not directly related to the pupil's thought-processes.)

The work of Rose (1977) gives several other examples, using a variety of problems with subjects of different ages and levels of mathematical competence - though almost all were chosen because they were available (!) and were also considered likely to be able to attempt the problems with fair success, and be able to vocalise fluently. To what extent similar activities, suitably transformed, with less-capable children would be productive of useful evidence of thought processes is the subject of other current studies.

A few general remarks, perhaps questions rather than statements, arise from the work done so far:-

The centrality of attitude is clear - willingness to guess and to formulate conjectures, which one may then prove incorrect, is obviously important. So also is the fear of failure.

In a discussion at the December 1978 meeting of the B.S.P.L.M., one participant remarked that children who are being encouraged to learn the skills of problem-solving must be prepared to meet "massive failure", and be insulated against it, for in such situations they will meet failure and insecurity more often than in conventional teaching. This seems a crucial point. Anxiety for success may make a pupil impatient - when it does not come quickly he becomes discouraged, and is not willing to persevere for an ultimate reward. The desire for visible signs of progress may be so overpowering that the problem is abandoned.

The role of checking previous work may well be related to this - results which have been verified as correct serve not only to prevent the pupil following mistaken paths, but by providing a firm platform for advance, help to relieve anxiety.

How useful is it for pupils to plan their work in terms of explicit strategies? This is a large question, but it does seem that this organisational capacity is an important factor in the work of successful students. Some are observed to announce their plans quite explicitly - this may be helpful in enabling them to concentrate on the details of current steps without getting lost within the problem as a whole.

What are the implications of such work? Clearly, if we aim to teach problem-solving, it will be helpful to us to know how problem-solving is carried out and what factors influence it. Two questions arise - how are we to teach problem-solving, and can it indeed be taught? Answers to these questions are someway off - these preliminary studies may at least suggest some possibilities which may usefully be explored.

Although Fischbein (1973) quotes evidence that explicit teaching of problem-solving skills is not effective, more recent work by Schoenfeld(1978) may indicate otherwise. In any case, the difficulty of investigating this issue is one of scale - problem-solving behaviour is a sizeable portion of an individual's mental activity - a large number of attitudes, skills and modes of thought are involved, and these are not likely to be susceptible to massive alteration by any short term "experimental" treatment of necessarily low intensity. Measurable effects seem likely only to be produced, if at all, by long exposure to appropriate training in problem-solving skills - to arrange this and to test for outcomes will not be easy.

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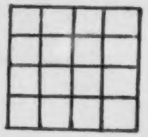
TRANSCRIPT 11/11/77, with Notes and Comments

well, first count ① ; I'll count the small squares as the most obvious ② to count first, ③ 16, er 4×4, er, the largest square probably the next ④ most easiest one, and then, starting ⑤ at.,. going larger from the smallest square, taking blocks of ⑥ four there's, erm, 4, 1 in each corner, ⑦ then there's two down the centre excluding the two blocks at either side, and can have a 2, and 2 if the board is ⑧ turned the other way round, erm; in 3s there are 4 of those, one at ⑨ each corner, each time excluding the seven outside them, so thats, erm, 16, 17, ...29 ⑩ so far, erm, the 4 by 4s ⑪ already been counted, erm, the... probably some more 4 by 4 ⑫ squares I should think, yes, those in the 4 by 4; oh dear! best start again; erm; ⑬ ⑭ 16 + 1, then there are 4 of the ⑮ 3 by 3 squares, and then the (⑯ 4 by 4) there's 1, 2, ,,,, 8, 9 of the small ones with 4 blocks in them, erm, ⑰ counted the ones with 3 blocks in them, counted the ones with 4, erm, with 2, (2 ⑱ by 2s) erm, yes I reckon thats 13, 14......30, actually ⑲, yes, 30; must have ㉓ missed one out before, just missed one out, probably must have made a mistake before, add 'em up ⑳ again; ㉑ 9 seems to have been the wrong thing; I didn't seem to add on the ㉒; yes, well one of them ㉔ is right! May as well move on to the second part. ㉕

Let's see: How many would there be ㉖ on a chessboard 8 by 8; must be a formula ㉗ for finding them, because ㉘ there's, for the 4 by 4 there's 1, for ㉙ ...... Call the largest ㉚ size x, largest size that's x, er 1 xs, for (x-1) there are ㉛, lets think what the relationship is, ㉜ well, going ㉝ increasing there's 16 small, 9 next, 4 next and 1 next ㉞ 16 is 4 squared, ㉟ 9 is 3 squared, 4 is 2 squared, and 1 is 1 squared; ㊱ so on a ㊲ chessboard, presumably, there ㊳ would be $8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1$, which equals 64 + 49 +. ㊴ + 4 + 1, erm adding them up: let's see ㊵ 9, 4, 1.. ㊶ ...add 9, 39 add 1, 40, another 4, 44 carry 4, oh dear ㊷ lost myself now, I'd better ㊸ start again; (adds up to ㊹ 204); so presumably, there would be, assuming the first ㊺ answer to be wrong, 30, ah, in the diagram of all sizes, and 204 on the chessboard ㊻ So the formula ㊼, probably ㊽, for finding it is, you take the largest number of squares ㊾ square that, x take ㊿ 1 squared, x take 2 squared, until the smallest number (51), smallest number (52), yes. Think I'll leave (53) that one, probably (54) finished.

How many squares are there in this diagram - of all sizes?



How many would there be on a chessboard (8 by 8) ?

THE PROBLEM

NOTES

1 plan
2 reason
3 reason
4 plan
5 system, order
6 not 2 by 2
7 reason
8 symmetry
9 system
10 completion ?
11 checking
12 uncertain
13 collapse of confidence - restarts
14 illogical order
15 system
16 inconsistent notation
17 check/recap
18 correct notation
19 notices discrepancy
20 explains
21 checks
22 pinpointing error
23 doesnt finish ?
24 cheerful - doesnt care
25 plan
26 reads Q
27 attitude
28 reason
29 detail
30 draws back from detail, notation
31 causing difficulty
32 too abstract
33 reverts to numbers
34 pattern
35 re-statement
36 pattern, push to extreme, confirmation
37 analogy
38 n = 8
39 satisfaction
40 routine
41 arithmetic
42 aware
43 plan
44 (detail omitted)
45 that is, 29
46 "answer the question"
47 general statement volunteered
48 doubt?
49 NOT using x
50 'x take 1' instead of minus.
51 =1 why not say so?
52 checks (yes)
53 plan
54 'probably' - open mindedness, or indecision.

A Final Project for 9th Grade

Development, Implementation and Analysis

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A final project, in the sense to be understood here, is an open-ended mathematical problem whose scope is wide *but strictly related to the regular curriculum.* The students can and are expected to apply a variety of the mathematical topics, techniques and mathematical thought processes from their experience to date. (Bruckheimer and Hershkowitz, 1977).

The principal goals we had in mind in developing such projects were:

1. to explore the level of mathematical thinking of students at the end of junior high school;
2. to enhance the mathematical maturity of 9th grade students both in the cognitive and affective domain;
3. as for goal number 2, but with the teacher as the target.

In this paper we concentrate on one project "Squares" as an example. We shall describe the stages of its development for various ability levels, its implementation and the analysis of student responses.

Development*

The project "Squares" was developed as a direct result of our wanting to create a situation in which the student "plays" with numbers, guesses and checks his guesses by applying the algebraic techniques in the curriculum. Further, we wanted a situation in which specification of the relevant domain (set of substitutions) has meaning; hence we chose a geometric form in which to pose the problem. Also such a setting would naturally invite further investigation of the expressions obtained.

We prepared an experimental version of the project and, as usual, tested it in teachers' workshops. From the work and discussion with the teachers we gained a clear impression that it was good material for a final project but would be too easy for more able students and too difficult for the weaker ones. Thus, we reworked the experimental version on the one hand for the weaker students by adding auxiliary sections and on the other hand, we enlarged the project for able students. It is worth mentioning that, in order to aid the teacher, we also developed a detailed teacher-guide in which we give several possible answers and the mathematical and didactical background describing the purpose and construction of the project.

Implementation

The final projects are designed for the students to do on their own over a period of about a month, with occasional "sign-posting" by the teacher. In spite of the positive reaction to the projects by the teachers not many of them actually implemented the projects on their own initiative. They were afraid that it would involve them in too much work and that they would not finish the regular program if they devoted time to the projects. Further, many of the teachers are mathematically very weak, and are frightened of open ended situations (see goal number 3 above). As part of a research project that started in 1978, we began to "push" things and to explore possible ways of assisting the teachers further in the implementation.

In the classes whose teachers agreed, we arranged two successive lessons and invited teachers from other classes and our own teacher guides to participate. In these two lessons the students were helped individually to get the feel for the style of work which was novel to them. Most of the time it was a pleasant experience for the students and the teachers. It was a kind of "mathematical happening". But there were difficulties, students "disturbed" each other with their "discoveries" and sometimes slower students were disappointed that they did not discover something on their own. In 1979, we used another method which facilitates research.

The students were asked to do the first part of each project at home without help and to bring their attempts to the teacher. Their work was checked and annotated by the teacher with our assistance. For each student we completed a check form (first stage). These forms gave us a picture of the class, described in the analysis section of this paper. Then a lesson was devoted to discussing the project, upon which, the students were asked to proceed and invited to get advice and help as they needed. They were usually hesitant to seek advice, because they somehow felt that it was then not their own work. ("How is it possible to give a mark for work if the teacher helps?") After a given period, they gave the project to the teacher for final checking and marking. Again for each student we completed a check form (final stage) dealing with the crucial sections of the project.

Analysis

In this section we describe some of the results on the project "Squares". Though it deals with a "routine" part of mathematics it involves various topics and has a certain richness. In the analysis we shall concentrate on results related to the concepts "open phrase" and "open sentence", which play a big role in our curriculum for Junior High School.

Population

The population which worked on the project was eight 9th grade classes. In each case they were A-stream (the highest ability level) classes, but since each school streams its own student population, there are significant variations in ability between schools. Therefore, four classes (n=81) were given the "easier" version "Squares II" and four classes (n=133) were given the "harder" version "Squares I".

**The development of curriculum materials in the science Teaching Department is always ateam effort. The major contributors to the development here dercribed were R. Hershkowitz, M. Bruckheimer and the author.*

Text of "Squares II"

Below we give the text of that part of the "Squares II" project relevant to the further analysis.

I (a) Draw a square with side of length 3 cm. Extend each side by 2 cm to obtain a new square. Colour and compute the additional area.

(b) Draw a square and denote the length of a side by x cm (x represents a positive integer). Extend each side by 2 cm to obtain a new square.

(c) (i) Write down an open phrase for the area of the square with side x cm.

(ii) Write down an open phrase for the area of the new enlarged square.

(iii) Write down an open phrase for the difference in area between the two squares (i.e. for the additional area) and reduce to simplest form.

(d) Compute the additional areas for five different integer values of x.

(e) Which of the following statements are true and which are false? (Justify your answers).

(i) The additional area is always represented by an odd number.

(ii) The additional area is always represented by an even number.

(iii) The additional area is always represented by a multiple of 4.

(iv) The additional area is always represented by a multiple of 8.

.

(h) If you know that the additional area is 128 cm^2, what can you say about the side of the original square?

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.
.

(k) If you know that the additional area is more than 114 cm^2, what can you say about the side of the original square?

.
.
.

(m) Try to pose a further problem, similar to those in the previous sections and answer your problem.

II. We are given a square whose side, measured in cm, is an integer and denoted by x. Each side is extended by a given amount to form a new square.

(a) The open phrase for the additional area is 5(2x+5). By how much was each side extended? Explain your answer.

.
.
.

(e) Write problems similar to those in sections (h) - (m) in Part I which apply to this case. Answer your problems.

III. Give a square whose length in cm is an integer denoted by x. By adding b cm to each side (b is a positive integer) a new square is obtained.

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.
.

(c) Given that the additional area is 175 cm^2, what can you say about the side length of the original square?

Discussion

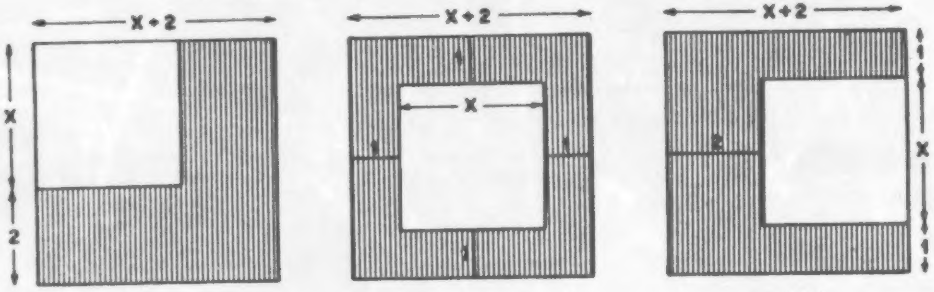
The construction of the question: we start from a concrete situation involving one variable, and get an open phrase, with which we "play" and then obtain some general conclusions about image sets for certain subsets of the domain. We then obtain corresponding open sentences and solve them, i.e. find subsets of the domain corresponding to given images. In the next stage we start with an open phrase and work back to the situation from which it arises, with a hint towards the two variable situation which comes later. Here is another opportunity to play; this time the added length is odd (5 cm) and not even (2 cm). In the third part we have an open phrase in two variables, more demanding on the student exploring this case.

Throughout the question we use phrases like "what can you say", indicating that we expect more than the usual brief answer. The project is opened up even further by the section in which we ask the students to invent similar problems of their own and to solve them (if possible), which is as close as we can get to real mathematical activity.

We shall now discuss the levels of mathematical ability as demonstrated in the students' answers.

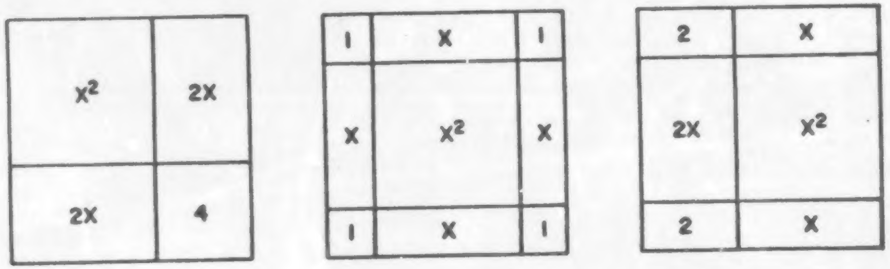
Part I (a)-(m) was given to the students in the first stage described above. Section (a) is very simple and was inserted merely to give a start and lead into section (b), where the question really begins. Most students managed to get started by themselves, except those of one class in which nearly all the students had no idea what they were being asked to do. These latter received some guidance in class so that they could continue with later sections.

There are several possibilities for the drawing; for example:



(i) (ii) (iii)

The vast majority drew a correct drawing; about half drew (i) and the other half (ii). There were a very few who drew something like (iii) or some other drawing or gave more than one correct drawing. Most students obtained the expression 4x+4 for the additional area and about 20% indicated relevant expressions on their drawing.



In section (d), about 20% of the students did not use the open phrase they had obtained previously but, each time computed the additional area directly. In general, these were the weakest students. In sections (h) and (k) one can use the relevant open sentences and answer the question algebraically. However, about 40% of the students (most already identified as being weak in earlier stages) found the answers by trial and error.

Section (d) was preparatory to section (e) which involves a generalization. An analysis of the responses gives the following distribution:

- did not answer correctly and did not explain	about 10%
- did not answer correctly but did explain	" 10%
- answered correctly with no explanation	" 30%
- explained correctly with numerical examples only	" 25%
- explained correctly using 4x+4	" 20%
- explained correctly using 4(x+1)	" 5%

It is worth mentioning that in one class (18 students) this project was given after they had done another one which contained a similar section. In this class only one student did not give any reason, three used examples only, and the remainder gave reasons, some of which were very elegant.

In section (m) about 50% did not answer at all. Most of those who did answer only changed the numbers of the previous sections. But there were some interesting original responses:

- *two limitations on the length of the side.*
- *" " " " additional area.*
- *a limitation on the side and a limitation on the area.*
- *attempts to identify numbers which could correspond to the additional area, using results from previous sections in the project.*

Again it is important to mention that, in the class that had previous experience of a final project, only one student did not answer this section and about half of the answers were interesting (original).

After the first stage, the students were given guidance and encouragement to continue, and they also consulted each other. Thus the number of responses of a certain type is not as relevant as the types of different responses obtained.

In section II (a) the responses demonstrate very clearly different levels of thinking:

- *first a guess that the extension is 5 cm and then a check that it really gives* $5(2x+5)$.
- *a general argument to obtain* $b(2x+b)$ *and then a claim that* b *needs to be 5 in the given case. (This was the most popular response).*
- *a general start followed by* $2bx+b^2 = 10x+25$, *and hence that* $b^2 = 25$, *i.e.* $b=5$ *and a check that for* $b=5$, $2bx = 10x$.

- *a backwards argument to obtain* $x^2+5(2x+5) = (x+5)^2$ *and hence that the extension is 5 cm.*
- *an interesting mistake:*

$$2bx + b^2 = 5(2x+5)$$
$$2bx - 10x = 25 - b^2$$
$$2x(b-5) = (5+b)\ (5-b)$$

if we divide by b-5 we get

$$b = -2x-5$$

since x is positive this is absurd and the student was embarrassed. There was a need for a discussion to get to the conclusion that, in fact, b=5 is the only solution.

Interesting responses were given by students who had learned the techniques of quadratic equation:

- *compare* $2bx + b^2 = 10x + 25$ *substituted 3 to obtain*

$$b^2 + 6b - 55 = 0$$

the positive solution is 5 and the negative one does not fit the problem. Some students tried more than one number to be sure that they "always" got 5.

- *general solution using a quadratic equation*

$$b(2x+b) = 5(2x+5)$$
$$b^2 + 2xb - 10x - 25 = 0$$
$$b_{1,2} = \frac{-2x \pm \sqrt{4(x+5)^2}}{2}$$

the negative root is not suitable because, even if b represents a decrease in side length, it exceeds x.

Therefore, only the positive solution is suitable:

$$b = \frac{-2x + 2(x+5)}{2} = 5$$

Part III takes us to a higher cognitive level:

Types of responses:

- $$b(2x+b) = 175$$
  thus $x = \dfrac{175 - b^2}{2b}$
- *b must be odd (based on previous sections)*

There were students who were not satisfied with these answers and wanted to find actual numbers.

An argument we heard went as follows:

- We need to find a value for b and a value for x.
- Perhaps there is more than one value. It says "What can you say ..."
- We had that in the previous section and, yet, we found just one number.
- That doesn't mean that the same thing will happen here ....

- *following Part II students wrote 5 for b and got* $x = 15$
- *by trial found some or all the possibilities for b;* 1, 5, 7
- *the full solution:*

Thus b can be either 1, 5, 7 or products of these.

for $b = 1$ $\quad 2x+1 = 175 \quad$ *thus* $\quad x = 87$
for $b = 5$ $\quad 2x+5 = 35 \quad$ *thus* $\quad x = 15$
for $b = 7$ $\quad 2x+7 = 25 \quad$ *thus* $\quad x = 9$

there is no point in going further, since $x > 0$ *and hence* $2x+b > b$.

This project is restricted to integers. There were suggestions from the students to remove this constraint, and we encouraged them. In fact, Squares I, intended for the stronger students, does precisely this. So we continue with a description of Squares I and the responses obtained.

Text of "Squares I"

Given a square with side of length x cm, a new square is obtained by adding 5 cm to each of its sides.

(a) Draw a suitable diagram. Shade the additional area (the difference between the areas of the two squares).

(b) Write down an open phrase for the additional area and reduce it to its simplest form. Specify the domain (set of substitutions) relevant to the problem.

(c) Show that the additional area (in cm^2) is an odd multiple of 5.

.
.
.

(f) Is it possible to have a situation in which the additional area is an integer multiple of the area of the original square?

(g) (i) If x is 10, what is the percentage increase in area?

(ii) Same question, but x is 50 cm?

(h) Find the side of the square for which the increase in area is closest to 100%?

.

(j) If instead of x an integer, we allow x to be any positive real number, discuss sections (b) - (i) accordingly.

II. What will happen if the side of the square decreases by 5 cm instead of increasing?

III. Now let's take a given square with side of length 4 cm and let the extension be a variable denoted by b.

.
.

IV. Let's modify our problem once again and look what happens if both the side x cm of the given square is a variable and the extension b cm, with the condition that b is proportional to x.

.
.

V. Let's explore a new problem: Given a square with side x cm. The length of one pair of opposite sides in increased by b cm and that of the other pair decreased by b cm, to obtain a rectangle.

.
.

VI. Try to invent more problems dealing with the relation between changes in length and in area.

Discussion

The students had no problem in beginning the project. In section (c) only 10% used numerical examples only to "prove" that the additional area is always an odd multiple of 5, about half used the phrase 10x+25 and the others factorised to obtain 5(2x+5).

Two of the four classes (n=64) had worked another final project before and none of these students relied on numerical examples only.

An interesting section is (f).

The only possibilities are $x = 1$, $x = 5$.

Distribution of responses:

- no response	about 20%
- impossible	" 5%
- possible without specifying values	" 5%
- found 1 and/or 5 by guessing and then checking (about half of these stated there are no more)	" 50%
- general but incomplete approach	" 10%
- complete (or almost complete) solution	" 10%

Here again the students that had had previous experience with a final project gave the best answers.

Types of response:

- $kx^2 = 10x + 25$

  $k = \frac{10}{x} + \frac{25}{x^2}$

  x^2 must divide 25 => x^2 is 1, 5 or 25

  x must divide 10 => x is 1, 2, 5 or 10

  Test to show that x is 1 or 5.

  Note - This response is incomplete, since one has to show that the sum cannot be an integer unless both fractional expressions are also integers.

- $k = \frac{5}{x}\left(\frac{2x+5}{x}\right)$

  One can see that x is only 1 or 5 (also incomplete).

- $k = \frac{(x+5)^2 - x^2}{x^2}$

  $k = \left(1+\frac{5}{x}\right)^2 - 1$

  k is an integer for $x = 1$ or $x = 5$ only.

- $kx^2 = 10x + 25$

  $x(kx-10) = 5 \cdot 5$

  thus x *can be only* 1 *or* 5 *(or* 25 *which fails)*

All the above are relatively sophisticated: a basically simpler type of response is given below. Just one student succeeded by this approach unaided. However, many attempted it this way and received guidance to complete

- $k = \frac{10x+25}{x^2}$

  Let's try to guess possible values for x.

  How many values are we going to need to check? Are we going to live long enough?

  Since $10x+25$ *is an odd number,* x^2 *and therefore* x *must be also odd.*

  What happens to k *as* x *increases? (*k *decreases)*

  When is $k < 1$*?* $(x > 13)$.

  So all we need check is $1 < x < 11$, x *odd.*

Responses given by students using quadratic equations

- $kx^2 - 10x + 25 = 0$

  $x = \frac{10 + 10\sqrt{1+k}}{2k}$ *(the positive root only is relevant)*

  Possible values of k *for which* $1+k$ *is a perfect square are:*

  3, 8, 15, 24, 35.

  (For larger numbers $x < 1$.) *Only 3 and 35 give* x *an integer.*

Section (g) was meant to surprise. In the previous sections the student noticed that the additional area increases with increasing x. Here the percentage *decreases* with increasing x. It is interesting to note once again that in the responses to this section there are indications that previous project experience enhances mathematical activity (see goal 2, page 4). Thus 9 of the students who had such previous experience, on their own initiative, with no leading question, remarked that the percentage decreases with increasing x. In the classes where there was no previous project experience, not a single student remarked on this fact.

In section (h) one can find the answer by guessing and it is always better to do this intelligently, i.e. by using the result of section (g).

Several students remarked that they found the answer (12) after a lot of trials, others constructed an organized table. One boy came up with the following:

"I found that for $x = 12$ the percentage is close to 100%. I want to know for what x it is exactly 100%. My brother told me that I need to solve a quadratic equation and I have not learnt that yet. Isn't there another way?"

Fortunately one of the girls in the class answered section (f) using

$$k = \left(1 + \frac{5}{x}\right)^2 - 1$$

We asked her to write it on the blackboard and asked the boy if he could answer his question now. He could!

Conclusion

The analysis of the responses to projects certainly does give us information on the levels of ability and mathematical thinking of students completing junior high school.

Further, there are clear indications that work on the projects has a cummulative and beneficial effect on the students' approach to and achievement in the mathematical activities. So we can say, at least intuitively and qualitatively, that the first two goals we set ourselves seem to be achievable to a significant extent. The projects also seem to be a very sound-basis on which to continue the type of research we have been doing with various refinements in design.

As far as the third goal is concerned, it will not have remained unnoticed that we said nothing. The teacher aspect has been disappointing. We shall have to give the teacher a lot more thought and attention.

The students, on the other hand, are highly motivated to work on projects and some produced most elegant and well organised work.

As a final word, we quote one student, who did a very nice piece of work on "Squares I". When he reached part V he was tired and wrote, "I will not do this part because I don't feel like doing it and I don't care what mark I get. Not because I don't know how to do it, just I don't want to. But I'll do part VI (the open question) because here I've several nice ideas."

We annotated this: "In a project like this, you don't have to do everything, and we agree that it is much more fun to work on the bits where one "has several nice ideas"... So why not?"

References

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Third Annual Meeting
of the
North American Chapter
of the
International Group for the Psychology of Mathematics Education

Thomas R. Post, University of Minnesota) Co-Chair
Sigrid Wagner, University of Georgia)

NEW DIRECTIONS IN THE PSYCHOLOGY OF MATHEMATICAL EDUCATION
A PERSONAL VIEW
WITH SPECIAL REFERENCE TO
THE ROLE OF P.M.E. / N.A.

Opening address
by
Richard R. Skemp, University of Warwick,
President, P.M.E.

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It is a great pleasure and honor to me that I have been given the opportunity to speak to you during our opening session. " /N.A." was not part of the original title; but as I came to write this talk, it became clear that this was what it was turning into. I hope that this revision is acceptable, but I feel that I owe you an explanation.

Within the past two years (slightly less) this is my fifth visit to your continent. (Three visits to the U.S.A., two to both Canada and the U.S.A.) From which you will be quite right in concluding that I like it here!

So I hope that some impressions of a friendly visitor from outside your continent may be of interest to this present audience. This leads me to emphasise the second part of my title: " ... a personal viewpoint". Between the time I received this invitation (July, at Grenoble) to now, there would have been no way I could have put together a properly balanced overview, based on a proper search of the literature. Fortunately we have Jeremy Kilpatrick's valuable review paper (Kilpatrick, 1981), less than 2 months old, which gives just such a survey.

I would like to talk about three topics: model building, teacher education, and computers.

MODEL BUILDING

I find it interesting and significant that Tom Romberg chose this area for his paper at Grenoble (Romberg, 1981), and particularly like his sentence (which I shall quote not only on this occasion): " ... the key step in (the) evolutionary sequence from myth and tradition to theory is model building." This is in the tradition of Dewey (1924), who wrote: "Theory is in the end ... the most practical of all things". (This sentence I was quoting for years before I read it, in paraphrase: not knowing that Dewey had said it first.) Kilpatrick also writes: "One of our greatest needs in research on mathematical learning and thinking is for conceptual theory-building analyses of the constructs and assumptions we are using." I don't agree with the view expressed by a prominent European maths educator* that what we need is not a global model but a collection of relatively specific models for specific situations. I think that we need both micro and macro-models. Even areas of mathematics which we regard as elementary, such as counting, addition, subtraction, reveal on investigation a conceptual complexity, a degree of interiority, which gives importance to the detailed investigations such as we have from Leslie Steffe (1981), Ernst Von Glasersfeld (1981), & John Richards (1981), and from Tom Carpenter, Jim Moser, and Tom Romberg (1982), to mention only some of those working in this particular area. The equality sign would seem to be an even smaller area, yet again we have micro-analyses of the ways in which children conceptualise its meaning from Carolyn Kieran (1980), from Merlyn Behr, Stanley Erlwanger, & Eugene Nichols (1980), and from Ed Labinowicz (1981), - again, I give only examples, not even the beginning of a listing. We certainly need these kinds of investigation, leading to micro-models giving much detail in small areas of children's mathematical learning.

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* In the spoken presentation I attributed this by name, and I am subjectively confident that I was correct. However, owing to imperfections in my filing system, I have not been able to locate the paper, and feel that it would therefore be improper to make the attribution in this printed version.

As an old schoolteacher of mine used to say, 'Many a mickle mak' a muckle', and the mathematical schemas which we want children to construct over a period of ten years or so are formed from many concepts, each of which has itself much interiority. Micro-models of this kind are needed not only to enable us to teach these concepts successfully, but also repeatedly to bring home to us, by a variety of examples, the great concentration of knowledge which mathematics contains. This is one of its strengths, but also a major difficulty in teaching, since it pre-disposes towards trying to teach too much, too quickly. The result is mental indigestion for the pupils (I have a reason for using this term rather than 'students' which there isn't time to pursue now), leading to serious misconceptions about the nature of mathematics itself. Instead of finding it a source of mental power, it becomes at best a chore and at worst a threat.

But we do also need a macro-model within which all these micro-models will eventually fit together, and for which at any particular point, one of these micro-models provides interiority. I see two complementary ways in which this may come about.

The first is the provision of a macro-framework, relating meaningfully the many aspects of mathematical education: cognitive, emotional, interpersonal, and also the goals of learners, the goals of teachers, parents, taxpayers, employers. Yes, I did say macro! But one of the things which my own model emphasizes (Skemp, 1979: this is a macro model) is that intelligent co-operation requires (among other things) compatible goals, giving rise to complementary plans. So we need to look at what these goals are, or we'll get (I would say we've got) mis-matches, failures of co-operation.

The second is by a mutual assimilation of these micro-models. Looking at the papers contributed by P.M.E. members over the past few years, I see good hope of this. As a contribution to helping to come about, I would like to make explicit what I see as four requirements, necessary but not sufficient, for mutual compatibility of researches. These are:

Models are of the same type.

Researchers have similar world views.

Researchers have similar goals

Researchers meet and communicate.

Models are of the same type. Here I refer to the distinction between type 1 models and type 2 models (Skemp, 1982). Type 1 models are models of our physical environment, such as are used in chemistry, electronics, geology. Type 2 models are models of how type 1 models are constructed and put to use. Type 1 models are of course mental models - conceptual models embodying regularities of the physical world. Type 2 models are mental models of how these conceptual structures are formed and used. They are mental models of the model-making-and-using process. Constructivist models are type 2 models. (Not the only kind. To give an example in quite a different universe, I would see Freudian theory as an early type 2 model.) If this distinction is accepted, it seems a necessary consequence that models of the same type can be mutually assimilated, but models of different types cannot. Another consequence is that the methodologies appropriate for the construction (building and testing) of type 1 theories may be quite inappropriate for type 2 theories. So people whose conception of what is an appropriate type of model for their own particular subject (in this case mathematical education) are respectively type 1 and type 2 will also disagree about what are acceptable criteria for the professional respectability of a piece of research. This disagreement will be especially likely to be a fruitless one, if they don't understand its source, and so cannot address themselves to the main point at issue. Those whose conceptions are alike (even if unconsciously so) will even when they disagree be more likely to do so fruitfully. That is, their discussions will have a good chance of advancing the thinking of both parties. (I can remember many discussions of the latter kind, as I'm sure most of us can.) This brings me directly to the second requirement for mutual compatibility of research models and methodologies:

Researchers have similar world views. For my own introduction to this concept, I am indebted to a paper by Patrick Thompson with the delightful title 'Were lions to speak, we wouldn't understand'. This paper describes so well the failure of communication between those holding different and incompatible world views that I won't spoil it by trying to précis it. I'll just recommend it for your reading, and add two postscripts of my own.

The first is that as I see it, the two camps which he describes are on the one side, of those who believe that all scientific theories are, or should be, of type 1; and on the other, of those who consider that theories of the kind we need for the improvement of mathematical education are of a different kind, that which I would call type 2 (though neither they nor Thompson are so far using these terms).

The second is that those who are as yet not fully committed will have a better chance of making the right decision which camp to join, given their particular universe of enquiry, if they clearly understand this distinction. And the third is that those who are already in the right camp may find this a useful argument in support of their viewpoint, and for encouraging others to share it.

The third requirement which I listed is that:

Researchers have similar goals. Elsewhere I have emphasised that we need, and try to construct, different models for different purposes. It's rather obvious that different models are needed for designing television sets and for designing bridges. It's less obvious that people with different goals will need appreciately different models even within the same subject matter. A colleague of mind at Warwick, Rolph Schwarzenberger (1986), came under fire recently for suggesting that engineers needed to learn calculus somewhat differently from pure mathematicians: but I agree with him. Among the goals of many* pure mathematicians, academic respectibility - achieving the maximum immunity to criticism, rigour at all costs including ease of understanding - come high. If someone had told me this at the beginning of my maths degree course at Oxford, I might have had a better chance of understanding, for example, not only how Dedekind sections worked but why I was required to learn them. I don't believe engineers need mathematical rigour of this kind. The sort of thing they need to know is quite different: for instance the tolerancies: how far the physical structure can differ from the designed model before stresses escalate. Lack of this elaboration of their model resulted in the collapse during construction of the Milford Haven Bridge, within sight of where I write this, at a cost of six lives and millions of pounds.

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* not all. It was G. H. Hardy who wrote 'Sometimes it is necessary to say difficult things, but they should be said as simply as possible'. (Hardy, 1967.) My own tutor at Oxford, W. L. Ferrar was another such example.

The point I have been leading up to is that maths education have different goals from mathematicians. For this reason, the reforms of mathematical education which were based largely on the models of mathematicians led to failure which in my view was predictable. Maths educators should not accept responsibility for this failure, and need to resist the backwash effect resulting from disillusionment at these failures. To give them confidence for this, they need, as Bauersfeld (1979) wrote to develop their own self-concepts. A shared theoretical position provides an excellent foundation for group solidarity: and this provides another example of 'There's nothing so practical as a good theory'. The co-operative move towards such a shared position I see as an important role for P.M.E.

Researchers meet and communicate. This is fairly obvious, so I can be brief. Nevertheless, it is non-trivial. During my last fly-about, I discovered (for example) that a group at Emory University, Atlanta, a doctoral student at Georgia University, and another group at the University of Texas, were all working on relational and instrumental understanding, without knowing about each other. I also discovered that North America is a large continent (it doesn't show like that in my atlas, because it is contracted on to a page of the same size as England), which partly explains why. Meetings such as the present one serve a vital function in bringing about contacts; and also the to-and-fro of discussion, which is so fruitful in sparking off new ideas, consolidating existing ones, and gradually knitting individual viewpoints together. Communication also takes place through the Proceedings of P.M.E. These constitute, in my own view one of the richest, most stimulating, and concentrated sources of ideas available in this field.

My next main heading is

TEACHER EDUCATION

Time is getting on, and I shall have to be much briefer on my last two topics. This will be easier because much of the earlier discussion is relevant here too.

For example, we can distinguish different world views held by teachers. One is that children don't know much, and it is our job to tell them and theirs to learn. Another is that children come into the world equipped with the best built-in learning ability of any species on this planet, and it is our job to co-operate with this and give it all the help we can. This may include telling them, but there are important other ways besides.

These world views are I believe subtly related to preferences for type 1 theories (exemplified by behaviourism) and type 2 theories (exemplified by, but not confined to, constructivism). Type 1 theories enable us to change the physical environment to our own ends: to make it behave the way we want. Applied to humans, the approach remains the same: we have terms like 'behaviour modification', and even 'human engineering'. The teachers' goals are to get children to behave rightly (which means, in ways approved by teachers). So what gets taught are rules for behaviour. Maths has the advantage that it is clear when rules have been kept, or broken.

Type 2 theories are mainly concerned with the building up of the mental resources from which right behaviour arises: conceptual structures, plans of action. (I say 'mainly, because skill in executive plans is important too.) So far as teachers are concerned, it is much less direct, and less directable. So far as learners are concerned, it is more direct because it is going on in their own minds under their own volition. So type 2 theories are more congenial to teachers whose approach to children is developmental.

Here I need to backtrack for a moment, to some impressions I formed when visiting the 1980 NCTM, at Seattle. Again, I must emphasise that this is a personal view, and based on a limited sampling. But I was disappointed and dismayed by my visit to the text-books exhibition: disappointed by the books, dismayed by the influence they would have on classroom practice. Again and again, my eye was caught by titles and presentations suggesting that I would find genuine mathematics, interestingly taught. But looking closer I found first some delightful pictures and chat about (e.g.) children buying cookies for a party: followed by the fateful words, 'Now do these': the rest of that page, and the next, consisting of mechanical arithmetic.

The two attitudes I've been describing can also be found transposed up a level, in implicit assumptions about teachers. One is that they aren't very bright, and so we've got to tell them exactly what to do or they'll get it wrong. The other is that they are fellow-professionals, doing a job complementary to our own and in some ways more difficult. The former attitude is implied by many teachers' guides: I am grateful to Tom Romberg for sending me the following example. (Engelmann and Carnine, 19 .)

How to point:

> Hold your finger about an inch from the page (or the board), just below the symbol.
>
> Be careful not to cover the symbol - all the children must be able to see it.
>
> Say, "Get ready." (In some formats, "Get ready" is replaced with a question such as, "What's this?")
>
> Hold your finger in the pointing position for at least one second.
>
> Look at the symbol to demonstrate to the children that they too should be looking at it.

The image of teachers which this kind of approach implies, in my view, is little short of insulting.

Nevertheless we should heed Kilpatrick's words (1981,p27) that " ... we still need to reconceptualise our approach so that mathematics teachers can become full partners in the research enterprise." We have to consider whether the models developed by researchers, in their full detail and complexity, are suitable in their original forms for communication to and use by teachers, or whether they may need to be re-stated. And if so, how can this be done without diminishing their power? This question was first brought to my notice last March when I visited Jacques Bergeron and Nicolas Herscovics in Montreal, who with Jean Dionne and Nicole Nantais-Martin are working on the communication to teachers of models of understanding. Shortly afterwards, at the University of Georgia, Tom Cooney pointed out to me that we need models of teachers as well as of learners. My current Primary Mathematics Project, now entering its fourth year, is classroom-based and in close collaboration with teachers: and this problem of re-stating a model without diminishing its power is currently of great concern to me.

So I think it is important that in P.M.E. we make it clear that we as researchers regard teachers as our natural allies in the improvement of mathematical education; and try to develop conceptual interfaces, possibly also organisational ones, whereby we can better learn from each other.

COMPUTERS

The latest arrivals on the educational scene are, of course, microcomputers. I am myself a newcomer to this field, having acquired one of my own only about two months ago. But I am already convinced that here is an influence we must not ignore, and find it surprising* that no paper in the P.M.E. proceedings for this or earlier years is about the use of computers in mathematical education. Programmed learning came and went, partly because of lack of sophistication in the hardware and partly because it was based on an inadequate learning model. I do not expect a similar departure of microcomputers from the educational scene. Month by month the hardware becomes more sophisticated, less expensive, and capable of supporting a steadily improving volume and quality of software. Though the present volume of software is small, it is on the increase: but the quality - oh, dear! Again, this is a personal impression, not a systematic survey, and if I have missed something good I hope that someone will tell me. But at a recent trade exhibition, by far the largest stand was that of Apple, and I watched a teacher trying out a maths programme from their range. On the screen was a division of a three or four digit number by twelve, the answer to be entered by the pupil. The teacher entered a wrong answer, no doubt intentionally. On the screen came 'No, that is not correct. Try again'. As I would have done in his place, the teacher again entered a wrong answer. 'That is still not correct. Think real hard.'

I have by me as I write the September 1981 issue of an English journal, "Educational Computing". Under the heading "Mathematics Special" is a collection of four articles about the use of computers for teaching mathematics. The first is a drill and practice programme for addition, in which (and I quote) 'The final act in the process is the drawing of a tick to show that the solution is correct'. The second is called "Consider the Binomial Theorem", and presents a programme for calculating the expansion of $(a + x)^n$. All the pupil has to do is to input numerical values of a, x, n. Why the coefficients are what they are was not discussed: and the article does not mention what the run-up to the use of this programme might be. In one way I liked this article somewhat better - it did at least show awareness that one job of a computer is to relieve the user of doing arithmetical lengthy calculations. But what I would really like to see would be a programme showing the combinatorial processes by which the terms and their coefficients are derived.

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* This statement was true when I wrote it, but on arrival in Minneapolis for the present meeting I found that Suzanne Damerin had contributed to the proceedings a paper expressing views which, though reached independently, are in close accord with my own.

The next article was a thoughtful one entitled "Simulation and sixth-form maths." The fourth describes the development of programmes " ... for the four rules of number." Back to drill.

The immense potential of microcomputers will be wasted if their use is based on teaching styles which belong to the past rather than the future, and on limited conceptions of the nature of mathematics. So may I offer two suggestions, diffidently as befits a newcomer to this field?

The first is a very simple one. May I warmly recommend Papert's book (Papert, 1981) 'Mindstorms: Children, Computers, and Powerful Ideas'. Apart from the title (storms for me suggest undirected energy) I like almost everything I have read in this. There do seem to be some practical limitations: 'Turtle' is not (anyhow as yet) a widely available language, it uses a lot of memory, and what it teaches is at present somewhat restricted. But what it does do, it does beautifully. And what I am getting out of Papert's book is a feeling for the relationship we should be aiming at between student and computer. (The switch from 'pupil' to 'student' is intentional, but there is not time to explain it here.) I am grateful to Papert for pointing me in what I think is the right direction for my own explorations.

At Berkeley last summer I watched a number of children using microcomputers, including those using the 'Turtle' language, and was greatly struck by the way in which they captured children's interest. It was quite hard for their parents to drag them away: one had to resort to bribery. I have since experienced for myself their addictive quality, and as a psychologist I am asking myself why this was so. Here is my suggestion, and I hope you will find it as intriguing as I do.

The microcomputer embodies, in microcosm, those qualities of the physical universe on which our human intelligence particularly likes to go to work. By constructing mental models of our environment, and devising plans of action based on these models, we can to an ever-increasing extent make the physical world do what we want. The long-term goal of the natural sciences is to comprehend the universe: but this is an ever-receding goal, and an individual in his lifetime can only take a few steps towards it. To understand a microcomputer, however, is a much more nearly attainable goal. What this object does it highly predictable, if (and only if) we have an appropriate mental model. By using this to devise suitable plans (programs), we can make the computer do what we want, to an ever-increasing extent and in increasingly complex ways. Thus the computer can be regarded as a particularly rewarding operand for the natural functioning of intelligence: and this I believe to be a major source of its attraction.

How can we harness this quality of microcomputer for the teaching of mathematics? The principle I believe is simple, though its application may take longer.

The model we have to build to achieve control of a particular computer is a model of the computer language and its syntax. When we have such a model, we can use it only for a limited - though important - category of physical objects, namely all computers which use that language. In this respect, knowing a computer language is less powerful than almost any branch of mathematics. One of the most important features of mathematics is that it is such a multi-purpose model, usable for a wide variety of features of the physical environment. So how would it be if we could arrange matters so that the models a student had to construct, to make the computer do what he wanted, were mathematical models? This is what Papert has done in a particular context. How far can we expand this?

If by the learning of mathematical models a child develops increasing mastery over a computer, these models will also develop his mastery over a wide variety of other objects in the physical environment. And he is acquiring these models in a way which gives him immediate experience of the same kind of power as mathematics confers in the realms of science, technology, and economics.

So, although some of the ways in which people are using computers for teaching mathematics are depressing, the possibilities are exciting. And who better to actualise these possibilities than members of P.M.E.?

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These, then, are the three major fields which I see as being, or needing to be, opened up: the co-operative construction of powerful, because appropriate, theoretical models for the teaching of mathematics; the development of conceptual and organisational lines of communication between researchers and teachers; and the actualisation of the immense potential of the microcomputer.

I believe that P.M.E. is rapidly becoming, perhaps already is, that group of people who are best able to influence mathematical education in those directions in which, for the sake of our children and their future, it needs to go.

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